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NON-ISOLATED HYPERSURFACE SINGULARITIES

Theo de Jong

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aan mijn ouders

CONTENTS

Over the last few decades there has been a lot of activity in the branch of mathematics called singularity theory. Here by singularities we mean either singularities of analytic spaces, or singularities of maps $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C}^p,0)$. The simplest case is the case of isolated hypersurface singularities. These are defined by a single germ of a holomorphic function $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$, such that 0 is an isolated singular point. Usually one puts some kind of equivalence relation on the set of all germs of holomorphic functions. We mention $\mathcal{R}$ (right) and $\mathcal{K}$ (contact) equivalence. Two germs f and g are called $\mathcal{R}$-equivalent if there exists a biholomorphic map $h\colon (\mathbb{C}^{n+1},0) \xrightarrow{\sim} (\mathbb{C}^{n+1},0)$ such that $f = g\circ h$; $\mathcal{K}$-equivalent if $f^{-1}(0)$ and $g^{-1}(0)$ are isomorphic as analytic spaces.

The ultimate goal in singularity theory is classifying singularities (say up to $\mathcal{R}$-equivalence). Here by classification we mean that we give a representative of each right equivalence class (normal forms), and methods to determine whether two functions belong to the same equivalence class. As with most general classification problems in mathematics this problem is much too difficult even to have a chance to be solved. One of the first classification results was obtained by Thom, who classified the so-called elementary catastrophes. Later Arnol'd made a "beginning" of the classification. To explain the word beginning, one introduces a useful invariant of a singularity called modality. In vague terms, modality is the number of continuous parameters occuring in a normal form of the singularity. For instance, the 0-modal singularities (also called simple singularities etc.) are classified by the well-known A-D-E list:

$$A_k\colon\ f = x_0^{k+1} + x_1^2 + \dots + x_n^2 \qquad k\geq 1$$

$$D_k\colon\ f = x_0^{k-1} + x_1^2x_0 + x_2^2 + \dots + x_n^2 \qquad k\geq 4$$

$$E_6\colon\ f = x_0^4 + x_1^3 + x_2^2 + \dots + x_n^2$$

$$E_7\colon\ f = x_1x_0^3 + x_1^3 + x_2^2 + \dots + x_n^2$$

E_8: $f = x_0^5 + x_1^3 + x_2^2 + \ldots + x_n^2$

Another notion which reflects very well the hierarchy of singularities is that of adjacency. A class of singularities K is called adjacent to another class of singularities L (notation $L \longrightarrow K$) if every function $f \in L$ can be deformed into a function of K by an arbitrarily small peturbation. Note that the adjacency relation defines a partial ordering on the equivalence classes of singularities. All adjacencies of the A-D-E singularities are given by:

$A_k \longrightarrow A_{k-1}$ $k \geq 2$,
$D_k \longrightarrow D_{k-1}$ $k \geq 5$, $D_k \longrightarrow A_{k-1}$ $k \geq 4$
$E_k \longrightarrow E_{k-1}$ $k=7,8$, $E_k \longrightarrow D_{k-1}$, $E_k \longrightarrow A_{k-1}$ $k=6,7,8$.

Both for classification and adjacency problems one introduces invariants of an equivalence class of singularities. One would like these to be easily calculable, take as much as possible different values on different equivalence classes and respect the partial ordering given by the adjacency relation in some sense. Many invariants stem from the so-called Milnor fibration. We give some examples, but we first recall how the Milnor fibration is defined (see [Mi]). Let $f:(\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a germ of a holomorphic function. Take $0 < \eta \ll \varepsilon$ sufficiently small. Let B_ε be the ball of radius ε in $\mathbb{C}^{n+1}$, and S^*_{η} the punctured disc of radius η. Then the map $f\colon B_\varepsilon \backslash f^{-1}(S^*_\eta) \longrightarrow S^*_\eta$ is a locally trivial fibration. A typical fibre F of this fibration is called the Milnor fibre. For an isolated singularity, F is homotopy equivalent to a bouquet of n-spheres; the number of these is called the Milnor number μ. This number is also equal to $\dim_{\mathbb{C}}(\mathcal{O}_{n+1}/J(f))$, $J(f)$ being the ideal generated by the partial derivatives of f. So the Milnor fibre has only one non-trivial reduced cohomology group $\tilde{H}^n(F,\mathbb{Z}) \cong \mathbb{Z}^\mu$. The group $H^n(F,\mathbb{Z})$ together with the intersection form on it (defined by intersecting cycles on the Milnor fibre) gives rise to the Milnor lattice. Also the monodromy leads to useful invariants . Walking once around the origin of the punctured disc S^*_η gives a diffeomorphism of the Milnor fibre, hence also an automorphism on the cohomology

$H^n(F,\mathbb{C})$. By the monodromy theorem [Br] the eigenvalues of the monodromy automorphism are roots of unity. A more detailed study gives rise to the notion of the spectrum of an isolated hypersurface singularity, which is a very useful and strong invariant. We refer the reader to [Lo], [A-G-V], [Di] for much more information on isolated singularities.

If one looks at the A-D-E list of singularities one sees two infinite series A_k and D_k occur. Also for the 1-modal and 2-modal singularities there appear chains of singularities which one would like to call series. Arnol'd recognized this phenomenon, and made some vague remarks about it (see [A-G-V]). Even now, a general definition of a series of singularities is not known. As Arnol'd says: "It is only clear that the series are associated with singularities of infinite multiplicity, so that the hierarchy of series reflects the hierarchy of non-isolated singularities". Indeed, deleting the part in the equations which varies with the indices one gets a function which one would like to call the stem of the series [Pe 2]. For instance for the A_k and D_k series one gets:

$$A_\infty\colon\ f = x_1^2 + \dots + x_n^2$$
$$D_\infty\colon\ f = x_1^2x_0 + x_2^2 + \dots + x_n^2.$$

Motivated by these remarks, Siersma [Si] started to study the simplest classes of non-isolated singularities, the so-called isolated line singularities. These are singularities with a smooth one dimensional singular locus and such that at a general point of the singular locus the germ is equivalent to an A_∞ singularity. Pellikaan, in his thesis [Pe 1], investigated hypersurface singularities with a general one dimensional singular locus. He obtained the best results if the singular locus is a one dimensional complete intersection singularity. Another point of view has been taken by Van Straten, who studied non-isolated singularities by means of an improvement. This is a good substitute for the notion of resolution for isolated singularities. Other investigations were made by Lê, Iomdin, Kato and Matsumoto and others, who were mainly concerned with the topology of the Milnor fibre.

This thesis is a continuation of the study of non-isolated singularities. In chapter one we study singularities with a smooth one dimensional singular locus Σ, which we call line singularities. Our purpose is to determine the homotopy type of the Milnor fibre. To describe the classes of line singularities for which we succeeded to solve this problem we introduce the notion of transverse singularity type on a branch of the singular locus of a non-isolated hypersurface singularity [Pe 1]. This is the μ-class (for the definition of μ-class see [Pe 1] I (9.1)) of singularities which one obtains by restricting the function to a transversal slice on a general point of the branch of the singular locus. If the transverse type is a simple singularity, then it is even defined as a $\mathcal{R}$-equivalence class. The classes we study in chapter one are line singularities with transverse type A_1, A_2, A_3, D_4, E_6, E_7 and E_8. The method we use consists of determining deformations which fix the singular locus and deform the transverse singularity trivially. We write down deformations such that only "elementary" singularities occur and maybe some A_1 singularities outside the singular locus. Now the Milnor fibre of the original function is built up out of the Milnor fibres of the elementary singularities and the A_1 points. Moreover algebraic formulas are given for the number of A_1 points and elementary line singularities occuring in the constructed deformation. For the number of A_1 points the formula is still conjectural in a number of cases. Moreover we remark that these formulas are only valid for the constructed deformations; there are counterexamples for other deformations. The main result is that in all cases we consider the Milnor fibre is homotopy equivalent to a wedge of n-spheres wedged with a wedge of (n-1)-spheres.

In chapter two we consider hypersurface singularities f with a one dimensional singular locus Σ and transverse type A_1. In case Σ is a complete intersection, Pellikaan proved that the only singularities one has to allow in a generic peturbation (in which it is assumed that the singular locus deforms in a flat way) are the A_∞, D_∞ and A_1 singularities. He also gives formulas in that case for the number of A_1 and D_∞ singularities. If I is the ideal defining the singular locus and J(f) the ideal generated by the partial derivatives he proves that:

$$\# A_1 + \# D_\infty = \dim_{\mathbb{C}}(I/J(f)).$$

Moreover because Σ is a complete intersection one has that $f \in I^2$ [Pe 1]. So if $I = (g_1,\ldots,g_n)$ with $g_1,\ldots,g_n$ a regular sequence one can choose a symmetric matrix h_{ij} such that $f = \sum h_{ij}g_ig_j$. Pellikaan proves that:

$$\# D_\infty = \dim_{\mathbb{C}}(\mathcal{O}/(I+\det(h_{ij})).$$

In case Σ is not a complete intersection, it is in general not possible to write down a deformation for which in a general fibre only A_∞, D_∞ and A_1 singularities occur. Indeed the ordinary triple point, defined by the function $f = xyz$, is rigid for the deformation theory of non-isolated singularities. Nevertheless we define a natural generalization of the number of D_∞ points, the so called virtual number of D_∞ points of f, denoted by $VD_\infty(f)$. It is proved that the total virtual number of D_∞ points is constant in a deformation. We remark here, that the definition is not very well suited for concrete computations. For surfaces we prove in chapter three (written jointly with D. van Straten) a formula for the virtual number of D_∞ points which might be easier to compute.
A nice surprise is that the virtual number of D_∞ points can be negative. This is best illustrated by the beautiful example of Pellikaan: $f = (yz)^2 + (xz)^2 + (xy)^2$, for which he gives two different deformations, one exhibiting 6 D_∞ and one ordinary triple point, the second one giving 4 D_∞ points. Because the total virtual number of D_∞ points is constant, we get that the virtual number of D_∞ points of the ordinary triple point is equal to -2. Using the virtual number of D_∞ points we prove a formula for the Euler characteristic of the Milnor fibre, which is a generalization of the corresponding formula for the case that the singular locus is a complete intersection. In the appendix of chapter two (written jointly with A.J. de Jong) we show that the local invariant VD_∞ adds up to a global invariant of a divisor in a smooth manifold. This generalizes a classical formula about the number of ordinary singularities on a surface in $\mathbb{P}^3$.

In the final chapter we investigate the deformation theory of non-isolated singularities. A good example to keep in mind is the same example as above. Pellikaan computed the base space of the versal family of this singularity. It is the union of a smooth 6 dimensional space with a general smooth 4 dimensional space in a 7 dimensional vectorspace. A simple but important remark is that the normalization $\tilde{X}$ of the space X defined by $f = 0$ is the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$. As is well known, the base space of the semiuniversal deformation of $\tilde{X}$ has the same structure (up to a shift of dimension), see [Pi]. This is not a coincidence: we will prove that the deformation functor $\mathrm{Def}(\tilde{X} \longrightarrow X)$ ($\tilde{X}$ normal, $X \subset \mathbb{C}^3$ with transverse A_1 singularities) is naturally equivalent to the functor of the so called admissible deformations of the non-isolated singularity X.

We give formulas for the vectorspace T^1 of admissible deformations over $\mathbb{C}[\varepsilon]$. We identify a space T^2 to which obstructions for lifting deformations to a slightly bigger base space map. A formula for the T^1 of the surface singularity $\tilde{X}$ can be given in terms of X alone. Furthermore, a formula for the dimension of smoothing components of normal surface singularities is proved. As a final application, we determine the "essential" part of the base spaces of the semiuniversal deformation of all rational quadruple points, without even having determined equations for these singularities.

As a by-product of the study of the deformation theory of non-isolated singularities we define, if the singular locus is non-obstructed, a non degenerate symmetric bilinear form on the normal bundle of the singular locus called the Hessian. This Hessian is important for two reasons. The first one is to get the obstruction space T^2 mentioned above as small as possible. The second reason is that there is a close connection to the virtual number of D_∞ points of chapter two.

For surfaces in $\mathbb{C}^3$ there is also an "antisymmetric Hessian". The geometric meaning of this Hessian, is up to now unclear. Also the problem to define a (anti-) symmetric Hessian in general is still open. We hope to work on these problems in the near future.

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SOME CLASSES OF LINE SINGULARITIES

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§0 Introduction.

(0.1) In this paper we study germs of holomorphic functions $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ with the properties:

1) The critical set $\Sigma(f)$ of f is smooth and one dimensional.
2) The transversal singularity of f in points of $\Sigma(f) - 0$ is of fixed simple type.

We consider these functions under right equivalence. If the coordinates of $(\mathbb{C}^{n+1},0)$ are $x,y_1,\dots,y_n$ we can and will assume that $\Sigma(f) = \{y_1 = \dots y_n = 0\}$. Therefore we call these singularities line singularities (cf. [Si 1]). The second condition means that for all $c \neq o$ and sufficiently small the isolated singularity

$f_c\colon (\mathbb{C}^n,0) \longrightarrow (\mathbb{C},0);\ f_c(y) := f(c,y)$

is of fixed simple type, called the transversal singularity of f. The simple isolated singularities were classified by Arnol'd, and we recall the list (see [A-G-V]).

A_k: $f = y_1^{k+1} + y_2^2 + \dots + y_n^2;\ k \geq 1.$

D_k: $f = y_1^{k-1} + y_1y_2^2 + y_3^2 + \dots + y_n^2;\ k \geq 4.$

E_6: $f = y_1^4 + y_2^3 + y_3^2 + \dots + y_n^2.$

E_7: $f = y_1^3y_2 + y_2^3 + y_3^2 + \dots + y_n^2.$

E_8: $f = y_1^5 + y_2^3 + y_3^2 + \dots + y_n^2.$

(0.2) The main purpose of this paper is to study the Milnor fibre. The starting point is a lemma of Siersma ([Si 2]), which states that if we have a deformation of a line singularity which is locally trivial outside the origin (i.e. "nothing happens outside 0"), then the induced Milnor fibrations above a small circle around the origin in $\mathbb{C}$ are equivalent. For a precise formulation see (3.2). It is handy to construct deformations in which only "elementary" singularities occur, which are hopefully easier to study. However, to the author's knowledge, no general theory about such deformations is available.

(0.3) The condition that for a given f the singular locus contains $\{y = 0\}$ (we write y for $y_1, \ldots, y_n$) can be expressed by $f \in (y)^2$. (see [Si 1]). But the condition that f has a given transversal singularity is much more difficult (except for transversal A_1). It is given by complicated equations, and even to determine these equations seems to be impractical. Moreover, if we have a singularity with given transversal type, it is not easy in general to construct deformations which are locally trivial outside the origin. One has to be careful that one does not destroy the transversal structure. We use the following trick to construct deformations:

Determine vectorspaces $I(S) \subset (y)^2$, where S is a given transversal type with the following properties:

1) It is so small, that every $f \in I(S)$ has at least transversal type S.
2) It is so big that:
   a) Every line singularity with transversal type S has a right representative which lies in I(S).
   b) If $f, g \in I(S)$ are line singularities with transversal type S, and f and g are right equivalent, then there exists a $h: (\mathbb{C}^{n+1},0) \xrightarrow{\sim} (\mathbb{C}^{n+1},0)$ with $f \circ h = g$ and $\underline{h^*(I(S)) = I(S)}$.

Given a line singularity with transversal type S, we can assume by 2.a) that $f \in I(S)$. Then $f + tg$ with $g \in I(S)$ will be a deformation which is locally trivial outside the origin. Of course, conditions 1) and 2) are so strong, that one cannot hope to find vectorspaces $I(S)$ for every S. We prove however that we can find vectorspaces $I(S)$ (which turn out to be ideals!) if the transversal type is A_1, A_2, A_3, D_4, E_6 and, if $n = 2$, E_7 and E_8. We therefore concentrate in this paper on these transversal types.

(0.4) Almost all constructions in this paper will be done with the help of these ideals. We introduce the codimension and the codimension zero and one singularities are classified. It will be proved that the codimension of f is finite if and only if f has $\{y = 0\}$ as critical locus and the transversal type is S. Moreover, these codimension zero and one singularities are precisely the elementary singularities we have to allow in a generic deformation, except for a few isolated Morse (A_1) singularities outside $\{y = 0\}$. One can easily give algebraic invariants which calculate the number of codimension one singularities in such a deformation. For the number of A_1 points, we define the Jacobi number. However, we cannot prove everything what seems to be true about this Jacobi number (see §2).

(0.5) In §3 we determine the homotopy type of the line singularities we consider with elementary algebraic topology. We follow Siersma's treatment ([Si 2]) closely. To apply it we have to determine the topology of the Milnor fibre of the codimension one singularities. It will turn out that the Milnor fibre has the homotopy type of

$\bigvee_{\epsilon} S^{n-1} \vee \bigvee_{\mu+\epsilon} S^n$, where ϵ is zero most of the time.

(0.6) Line singularities with transversal type A_1 were studied by Siersma ([Si 1]). His results have been generalized to the case that the singular locus Σ is a one dimensional complete intersection, and the transversal type is A_1 by Pellikaan and Siersma ([Pe], [Si 2]). There are recent results for Σ not a complete intersection. We refer to the thesis of Pellikaan ([Pe]) and a preprint of the author ([Jo]).

(0.7) The coordinates of $\mathbf{C}^{n+1}$ will be x, $y_1,\ldots,y_n$. But if $n = 2$ we often use the notation $y_1 = y$, $y_2 = z$, and if $n = 3$ also $y_3 = w$. Sometimes we list only the function in the stable equivalence class with the minimal number of variables. Throughout the paper $n \geq 2$, and the singular locus $\Sigma = \{y_1 = \ldots = y_n = 0\}$. Moreover sometimes we use $\mathcal{O}$ instead of $\mathbf{C}\{x,y_1,\ldots,y_n\}$.
We refer to Siersma ([Si 2]) for references concerning non isolated singularities.

(0.8) Acknowledgements: The author wishes to thank R. Pellikaan, D. Siersma, J. Steenbrink and D. van Straten for many discussions and advices. Moreover thanks to the referee for comments on an earlier draft of this paper, and to the University of Kaiserslautern for its hospitality.

§1. Deformations and Invariants.

(1.1) In this paragraph we determine ideals $I(S)$, define invariants and prove a Morsification theorem.

(1.2) Proposition. Let $f: (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a line singularity and let the transversal type S be either A_1, A_2, A_3, D_4, E_6, E_7 or E_8. Assume $n = 2$ if $S = E_7$ or E_8. Then f is right equivalent to a germ $g \in I(S)$, where I(S) is defined as follows:

$I(S) = \{$ ideal of functions g with order(g) ≥ 1 $\}$, where the weights of the coordinates are: weight(x) = 0, weight $(y_i) = 1/2$ $(i = 3,\dots,n)$ and

S	weight(y_1)	weight(y_2)
A_1	1/2	1/2
A_2	1/3	1/2
A_3	1/4	1/2
D_4	1/3	1/3
E_6	1/4	1/3
E_7	2/9	1/3
E_8	1/5	1/3

Table 1

Proof: The proof is elementary. First of all, because $\Sigma(f) = (y)$, we have that $f \in (y)^2$. See Siersma ([Si 1]). This proves the result for $S = A_1$. Consider the 2-jet of f in $y_1,\dots,y_n$. This can be seen as a quadratic form in $y_1,\dots,y_n$ with coefficients in $\mathbb{C}\{x\}$. Because one can take the square root of a unit in $\mathbb{C}\{x\}$ and $1/2 \in \mathbb{C}\{x\}$, we can change the coordinates linearly such that the 2-jet has the form (see [M-H]):

$$(*) \quad \sum_{i=1}^{n} \alpha_i \cdot y_i^2 \qquad \text{with } \alpha_i \in \mathbb{C}\{x\}.$$

If $S = A_k$ $(k \geq 2)$ resp. D_k $(k \geq 4)$, E_6,E_7,E_8 then the corank of this quadratic form is 1 resp. 2. (See [A-G-V]). Therefore, after a permutation of the y_i, we may assume that $\alpha_1 = 0$ resp $\alpha_1 = \alpha_2 = 0$ in (*), proving the result for $S = A_2$ and D_4. If $S = A_3$, we have just seen that we may assume:

$$f = A\cdot y_1{}^3 + \sum_{i,j=2}^{n} C_{ij}y_iy_j + R,$$ order $(R) > 1$, where the weights of the coordinates are as in the $S = A_2$ case, and $A, C_{ij} \in \mathbb{C}\{x\}$. For $c \neq 0$ and sufficiently small we have that $\det (C_{ij}(c)) \neq 0$, because the quadratic form has corank 1. But then $A(c) = 0$, since otherwise the transversal type would be A_2. Thus $A = 0$, which completes the proof in case $S = A_3$. Now let $S = E_6$, E_7 or E_8. Because we have $\alpha_1 = \alpha_2 = 0$ in (*) we have:

$$f = A'y_1^3 + B'y_1^2y_2 + C'y_1y_2^2 + D'y_2^3 + \sum_{i=3}^{n} \alpha_i y_i^2 + R,$$ with $\mathrm{order}(R) > 1$, where the weights of the coordinates are as in the $S = D_4$ case and $A', B', C', D' \in \mathbb{C}\{x\}$. Then:

$$f = x^q\cdot(Ay_1^3 + By_1^2y_2 + Cy_1y_2^2 + Dy_2^3) + \sum_{i=3}^{n} \alpha_i y_i^2 + R;$$ $q \in \mathbb{Z}$, $q \geq 0$, and A, B, C, or D a unit in $\mathbb{C}\{x\}$. Therefore, after a possible interchange of y_1 and y_2 we have that C or D is a unit. For every $c \neq 0$, the 3-jet of $f_c(y) := f(c,y)$ has to be equivalent to $(\alpha y_1 + \beta y_2)^3$, because $S = E_6$, E_7 or E_8. (See [A-G-V]). So for every $c \neq 0$ the 2×2 minors of

$$\begin{bmatrix} A(c) & B(c) & C(c) \\ B(c) & C(c) & D(c) \end{bmatrix}$$ must be zero. Hence $AC = B^2$, $AD = BC$, $BD = C^2$. If C is a unit, then it follows that D is a unit, and we can perform a change of coordinates such that $D = 1$. Thus

$f \sim x^q(Cy_1 + y_2)^3 + R'$, and the following change of coordinates proves the proposition in case $S = E_6$: $x \longrightarrow x$, $y_i \longrightarrow y_i$ $(i \neq 2)$

$y_2 \longrightarrow y_2 - Cy_1$. Now assume that $n = 2$ and $S = E_7$ or E_8. Then the above argument shows that we may assume

$f = Ay_1^4 + By_1^3y_2 + Cy_1^2y_2^2 + Dy_2^3 + R$, $\mathrm{order}(R) > 1$, $\mathrm{weight}(y_1) = 1/4$, weight$(y_2)$ = 1/3, A, B, C and D $\in \mathbb{C}\{x\}$. Because f has transversal E_7 or E_8-singularities we have that $D \neq 0$. Therefore A = 0, because otherwise the transversal type would be E_6. This gives the result for $S = E_7$. For $S = E_8$ we may assume

$f = Ay_1^5 + By_1^3y_2 + Cy_1^2y_2^2 + Dy_2^3 + R$, order(R) > 1, weight$(y_1)$ = 2/9, weight(y_2) = 1/3, A, B, C and D $\in \mathbb{C}\{x\}$. Then it follows that B = 0. This concludes the proof of the proposition. □

Remark. Property 2.b of (0.3) will be proved in (1.7).

(1.3) Definition Let $f \in I(S)$ as in (1.2), and

$g \in I(S).\mathbb{C}\{x,y_1,\ldots,y_n,t\}$. Then f + tg will be called an I(S)- deformation of f.

(1.4) Let us indicate why we cannot find vectorspaces I(S) with the properties 1) and 2) for $S = A_4$, $S = D_5$, and $S = E_7$ if $n \geq 3$. For simplicity we take n = 2 and $S = A_4$. Then the analogue of the ideals I(S) as in Prop. (1.2) in the $S = A_4$ case is $I = (y^5,\ y^3z,\ z^2)$, consisting of the funcions of order ≥ 1, where weight(y) = 1/5, weight(z) = 1/2. We certainly need all the functions in this ideal because of the examples $z^2 + y^3z$, $x^\alpha z^2 + y^5 + z^\beta$, $\alpha,\beta \in \mathbb{N}$ etc. But remark that the 4-jet in y, z of an $f \in I$ is divisible by z, so in particular by reducibles. But in the examples

$f_\alpha = (x^\alpha z + y^2)^2 + y^5 + z^3$ the 4- jet in y, z is not reducible, so f_α is not equivalent to a germ in I. Now one can try to take $I + (f_\alpha)_{\alpha\in\mathbb{N}}$. But in this vectorspace there are funtions with transversal type A_3 (so it does not have property 1) of (0.3)), for example $f_1 + z^2$!

For $S = E_7$ and n = 3 one can take the "same" examples

$f_\alpha = (x^\alpha w + y^2)^2 + y^3 z + z^3 + y^5$.

For $S = D_5$ and $n = 2$ $f_\alpha = (x^\alpha y + z)^2\cdot(xy + z) + z^4 + y^4$. It is left to the reader to check that these functions have no representative in $(y^4,\ y^3z,\ yz^2,\ z^3)$. Of course, the search for vectorspaces for more complicated transversal singularities becomes hopeless.

(1.5) Next we want to introduce the codimension. Let $I \subset \mathbf{C}\{x,y_1,\dots,y_n\}$ be an ideal, and $\mathcal{D}$ be the group of local analytic isomorphisms $h: (\mathbf{C}^{n+1},0) \xrightarrow{\sim} (\mathbf{C}^{n+1},0)$.

__Definition.__ $\mathcal{D}_I = \{h \in \mathcal{D}: h^*(I) = I\}$.

We have a right action of $\mathcal{D}_I$ on I, and thus for every $f \in I$ the orbit $\mathcal{O}(f) \subset I$. Consider the tangent space $T\mathcal{D}_I$ of $\mathcal{D}_I$ at the identity. This tangent space is contained in the tangent space $T\mathcal{D}$ of $\mathcal{D}$ at the identity. We make the identification of $T\mathcal{D}$ with $m\cdot\theta$, the germs of vector fields which are zero at the origin. The proof of the following lemma can be found in Pellikaan ([Pe, pg.19]).

__Lemma.__ $T\mathcal{D}_I = \{\xi \in m\cdot\theta: \xi(I) \subset I\}$. □

__Definition.__ Let $f \in I$.

$$c(f) = c_I(f) = \dim_{\mathbf{C}} I/T\mathcal{D}_I(f),$$

where $T\mathcal{D}_I(f) = \{\ \xi(f): \xi \in T\mathcal{D}_I\ \}$.

We call c(f) the codimension of (the orbit of) f.

(1.6) The following finite determinacy theorem is useful for the classification of functions of low codimension. We state it without proof, because it is standard (cf. [Si 1]).

Definition. Let $I \subset \mathbb{C}\{x,y_1,\dots,y_n\}$ be an ideal. Then $f \in I$ is called k-determined in I ($k \in \mathbb{N}$) if $f + m^k \cdot I \subset O(f)$.

Theorem. Let $f \in I$. Then:

a) If f is k-determined in I then

$$I \cdot m^k \subset TD_I(f) + I \cdot m^{k+1}.$$

b) If $I \cdot m^k \subset m \cdot TD_I(f)$ then f is k-determined in I. □

Corollary: Let $f \in I(S)$ as in Prop. (1.2). Then

$$c_{I(S)}(f) < \infty \Leftrightarrow f \text{ is k-determined in } I(S) \text{ for some } k \in \mathbb{N}. \quad \square$$

(1.7) **Lemma.** Let $f \in I(S) = I$ be a line singularity with transversal type S as in Prop. (1.2), and $O(f)$ resp. $\mathrm{Orb}(f)$ be the orbits of f under the right action of D_I resp. D. Then:

$$I \cap \mathrm{Orb}(f) = O(f).$$

Proof: The inclusion $\supset$ is trivial. For the other inclusion, we follow Pellikaan ([Pe, pg. 37-39]). We only have to proof that $mJ(f) \cap I = TD_I(f)$. Let $x = y_0$ for notational convenience. We give the singular locus Σ the (in general) non-reduced structure of $I' = \{\, \partial/\partial y_i(h),\ i = 0,\dots,n,\ h \in I(S)\}$.
It is easily checked that Σ is Cohen-Macaulay. Let $I' = (g_1,\dots,g_l)$.
Then $\partial f/\partial y_i = \sum_k \varphi_{ik} g_k$, and $\partial^2 f/\partial y_i \partial y_j = \sum_k \varphi_{ik} \partial g_k/\partial y_j \bmod (I')$, and let $\varphi\colon O_\Sigma^l \longrightarrow O_\Sigma^{n+1}$ be the map with matrix (φ_{ik}). Consider the following commutative diagram:

$$\begin{tikzcd} O^{n+1} \arrow[d, "\overline{dg}"'] \arrow[dr, "\overline{d^2f}"] & \\ O_\Sigma^l \arrow[r, "\varphi"] & O_\Sigma^{n+1} \end{tikzcd}$$

Trivially $\mathrm{Ker}(\overline{d}^2\overline{f}) \supset \mathrm{Ker}(\overline{dg})$, and one checks, by a local computation o that $\mathrm{Ker}(\overline{d}^2\overline{f}) = \mathrm{Ker}(\overline{dg})$ outside zero. Now let $\xi \in \mathrm{Ker}(\overline{d}^2\overline{f})$. Then $\overline{dg}(\xi) \in \mathcal{O}^1_\Sigma$ is zero almost everywhere. But because we have given Σ a Cohen-Macaulay structure, it follows that $\overline{dg}(\xi) = 0$, and thus $\mathrm{Ker}(\overline{d}^2\overline{f}) = \mathrm{Ker}(\overline{dg})$. Now suppose that $\Psi \in m J(f) \cap I$. Then $\Psi = \sum_i \xi_i \partial f/\partial y_i$, with $\xi_i \in m$ and $\partial f/\partial y_i \in I'$. Then $\partial\Psi/\partial y_j = \sum_i \xi_i \partial^2 f/\partial y_i \partial y_j \equiv 0 \pmod{I'}$. Hence $(\xi_0,\ldots,\xi_n) \in \mathrm{Ker}(\overline{d}^2\overline{f}) = \mathrm{Ker}(\overline{dg})$. But $(\xi_0,\ldots,\xi_n) \in \mathrm{Ker}(\overline{dg})$ means that $\xi_0\partial/\partial y_0 + \ldots + \xi_n\partial/\partial y_n \in T\mathcal{D}_{I'}$. Now it is left to the reader to check that for each of the ideals $I(S) = I$ of Prop. (1.2) $T\mathcal{D}_{I'} = T\mathcal{D}_I$. Therefore $\Psi = \xi(f) \in T\mathcal{D}_I(f)$.□

Remark. The above lemma shows that every invariant of $f \in I(S)$ which we define with the help of the ideals $I(S)$ is in fact an invariant of the right equivalence class of f.

(1.8) For the ideals $I(S)$ as in prop.(1.2), the codimension zero singularities are easily classified. In fact, one takes the normal forms of S as in (0.1) and considers them as functions $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ with the extra variable x, i.e. "nothing happens at the origin". We denote the right equivalence class of this singularity by ΠS.

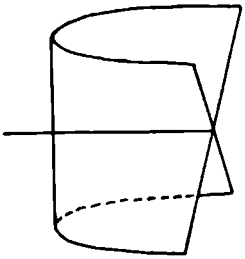
(1.9) <u>Theorem.</u> The following table gives the codimension one singularities for the ideals I(S) as in Prop. (1.2). Recall the conventions of (0.8).

S	F_1S	F_2S	F_3S
A_1	$xy^2 + z^2$		
A_2	$xy^3 + z^2$	$xz^2 + y^3$	
A_3	$xz^2 + y^2z$	$xy^4 + z^2$	
D_4	$xz^3 + y^2z$	$y^3 + z^3 + xw^2 + yzw$	
E_6	$xw^2 + y^2w + z^3$	$xy^4 + z^3 + y^3z$	$y^4 + xz^3 + y^2z^2$
E_7	$xz^3 + y^3z$	$xy^3z + z^3 + y^5$	
E_8	$xy^5 + z^3 + y^4z$	$y^5 + xz^3 + y^2z^2$	

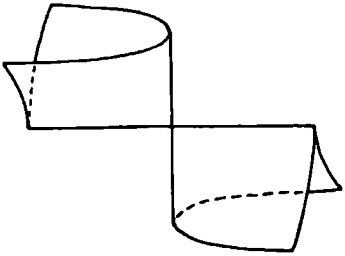
Table 2

We omit the proof.

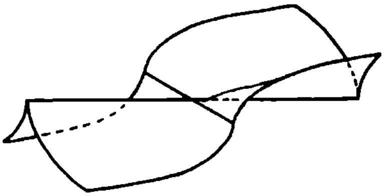
<u>Remark:</u> F_1A_1 is called D_∞ by Siersma ([Si 1]).



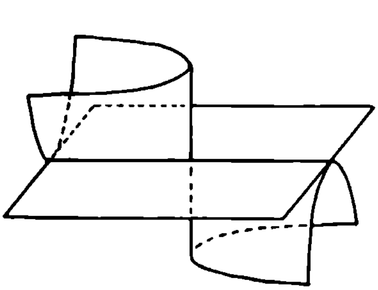
F_1A_1



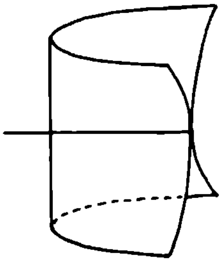
F_1A_2



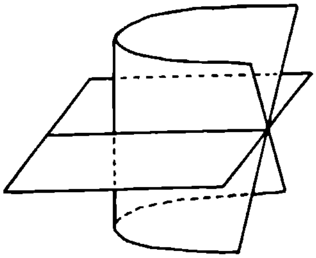
F_2A_2



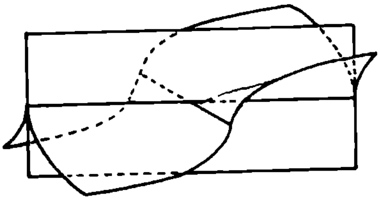
F_1A_3



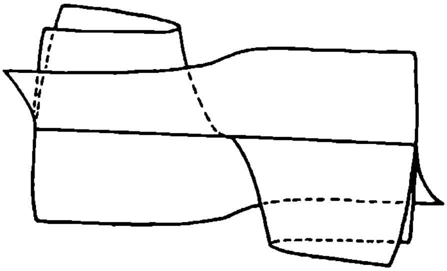
F_2A_3



F_1D_4



F_1E_7



F_2E_7

(1.10) We remark that from $S = D_4$ on, we see that some functions which are "expected" to be of codimension one, do not occur. For example, at first glance one would expect $f = xw^2 + y^3 + z^3$ to be a codimension one germ. Of course, one can easily calculate that $c(f) = 2$. Observe that f and our F_2D_4 are both homogeneous and therefore define cubic surfaces in $\mathbb{P}^3$ with one D_4 singularity. There are two types of these surfaces. One with an Eckart point (i.e. a point on the surface where three lines on the surface meet), and one without. Our F_2D_4 is the one without. Similar for $f = xz^3 + y^4$ and F_3E_6. As quartics in $\mathbb{P}^2$ f has two flexes, but F_2D_4 has only one.

(1.11) Our next invariant is the Jacobi number, which is important for the deformation theory (see §2). We need an auxiliary definition.

Definition 1. Let $I \subset \mathbb{C}\{x, y_1, \ldots, y_n\}$ be an ideal.

$\bar{\mathcal{D}}_I = \{h \in \mathcal{D}_I : h(x,y) = (x,y) \text{ for all } (x,y) \in V(\mathrm{rad}(I))\}$.

Definition 2. Let $f \in I$, with I as in definition 1.

$j(f) = j_I(f) = \dim_{\mathbb{C}} I/T\bar{\mathcal{D}}_I(f)$.

Remarks:

1) Because $\bar{\mathcal{D}}_I$ is an normal subgroup of $\mathcal{D}_I$, we have that every $h \in \mathcal{D}_I$ gives an ismorphism h_*: $T\bar{\mathcal{D}}_I(f) \longrightarrow T\tilde{\mathcal{D}}_I(f \circ h)$. This gives us the fact that the invariant $j_I(f)$ is an invariant of Orb(f).
2) $T\bar{\mathcal{D}}_I = \{\xi \in \mathrm{rad}(I) \cdot \theta : \xi(I) \subset I\}$.
3) In case $S = A_1$, Siersma ([Si 1]) defined the invariant j(f) in a different way. But because Siersma's j(f) and our j(f) both calculate the number of A_1 points plus the number of D_∞ points in a generic deformation (see (1.15), (2.1) and [Pe pg. 80]), we get that they are equal.

(1.12) Let $f \in I(S)$ as in Prop. (1.2), such that $c(f) < \infty$. Let $k \gg 0$, so that f is k-determined in I(S). Consider the map

$$j^k(f): \mathbb{C} \longrightarrow J^k I(S)$$

which sends $x \in \mathbb{C}$ to its k-jet $j^k_{(x,0)} f$ in $(x,0) \in \mathbb{C}^{n+1}$ in $J^k I(S)$. Let $Z_1, \ldots, Z_s$ be the closures of the codimension one orbits $F_i S$ in $J^k I(S)$.

__Definition.__ $h_i(f) = \langle j^k(f)\ (\mathbb{C}),\ Z_i\rangle$; $i = 1,\ldots,s$, where $\langle .,.\rangle$ denotes the intersection number (see e.g. [Fu]). This definition is independent of $k \gg 0$.

(1.13) One can work out definition (1.12) in the separate cases. This is not difficult. It turns out that $h_i(f) = \dim_{\mathbb{C}} \mathbb{C}\{x\}/(H_i f)$, where the $H_i f$ are defined as follows:

a) $S = A_1$: $f = \sum_{i,j=1}^{n} h_{ij}y_iy_j$; $h_{ij} = h_{ji}$. $H_1 f = \det h_{ij}(x,0)$.

b) $S = A_2$: $f = Ay_1^3 + \sum_{i=2}^{n} B_iy_1^2y_2 + \sum_{i,j=2}^{n} C_{ij}y_iy_j$; $C_{ij} = C_{ji}$.

$H_1 f = A(x,0)$; $H_2 f = \det C_{ij}(x,0)$.

c) $S = A_3$: $f = C_{11}y_1^4 + \sum_{i=2}^{n}(C_{1i} + C_{i1})y_1^2y_i + \sum_{i,j=2}^{n} C_{ij}y_iy_j$; $C_{ij} = C_{ji}$.

$H_1 F = \det C_{ij}(x,0)_{i,j\geq 2}$; $H_2 f = \det C_{ij}(x,0)$.

d) $S = D_4$: $f = Ay_1^3 + By_1^2y_2 + Cy_1y_2^2 + Dy_2^3 + \sum_{i,j=3}^{n} C_{ij}y_iy_j + \ldots$;

$C_{ij} = C_{ji}$, where the dots mean that we have not used all generators of $I(D_4)$, but these terms are not important for the $H_i f$. Let Δ be the discriminant of the polynomial $Az^3 + Bz^2 + Cz + D$.

$H_1 f = \Delta(x,0)$; $H_2 f = \det C_{ij}(x,0)$.

e) $S = E_6$: $f = Ay_2^3 + C_{11}y_1^4 + \sum_{i=3}^{n}(C_{1i} + C_{i1})y_1^2y_i + \sum_{i,j=3}^{n} C_{ij}y_iy_j + \ldots$;

$C_{ij} = C_{ji}$, and the dots have the same meaning as under d).

$H_1 f = \det C_{ij}(x,0)_{i,j\geq 3}$; $H_2 f = \det C_{ij}(x,0)$; $H_3 f = A(x,0)$.

f) $S = E_7$: $f = Ay^5 + By^3z + Cy^2z^2 + Dz^3$.

$H_1 f = D(x,0)$; $H_2 f = B(x,0)$.

g) $S = E_8$: $f = Ay^5 + By^4z + Cy^2z^2 + Dz^3$.

$H_1 f = A(x,0)$; $H_2 f = D(x,0)$.

(1.14) The following theorem can be seen as an analogue of the theorem that the algebraic Milnor number of f is finite iff f has an isolated singularity. For the moment, for $f \in I(S)$, the $h_i(f)$ are defined as in (1.13).

<u>Theorem.</u> Let $f \in I(S)$ as in Prop. (1.2). The following are equivalent:

1) f is a line singularity with transversal type S.

2) $c(f) < \infty$.

3) $j(f) < \infty$.

4) $\Sigma(f) = (y = 0)$ and $h_i(f) < \infty$ for all i.

<u>Proof:</u> Let $U \subset \mathbb{C}^{n+1}$ be a small neighbourhood of 0. Let $\mathcal{O}$ be the sheaf of holomorphic functions on U and $\mathcal{I}$ the sheaf of ideals such that $\mathcal{I}_{(x,0)} = I(S)$ and $\mathcal{I}_{(x,y)} = \mathbb{C}\{x,y_1,\dots,y_n\}$ for $y \neq 0$ and $(x,y) \in U$. Then $T\mathcal{D}_{\mathcal{I}}(f)$ and $T\tilde{\mathcal{D}}_{\mathcal{I}}(f)$ are sheaves of ideals too. We define sheaves of $\mathcal{O}$-modules. $\mathcal{F}^1 = \mathcal{I}/T\mathcal{D}_{\mathcal{I}}(f)$ and $\mathcal{F}^2 = \mathcal{I}/T\bar{\mathcal{D}}_{\mathcal{I}}(f)$. $\mathcal{F}^1$ is coherent as well as $\mathcal{F}^2$. We want to use:

$\mathcal{F}^i$ is concentrated in finite set of points $\Leftrightarrow \dim_{\mathbb{C}}\Gamma(\mathcal{F}^i) < \infty$.

2) $\Rightarrow$ 3) By definition $T\bar{\mathcal{D}}_{\mathcal{I}}(f) \subset T\mathcal{D}_{\mathcal{I}}(f)$ and $\mathfrak{m}T\mathcal{D}_{\mathcal{I}}(f) \subset T\bar{\mathcal{D}}_{\mathcal{I}}(f)$. Therefore $\mathcal{M} := T\mathcal{D}_{\mathcal{I}}(f)/T\bar{\mathcal{D}}_{\mathcal{I}}(f)$ is a finitely generated $\mathcal{O}$-module with $\mathfrak{m}\mathcal{M} = 0$. Hence $\dim_{\mathbb{C}}\mathcal{M} < \infty$ and thus $j(f) < \infty$.

3) $\Rightarrow$ 1) $j(f) < \infty$, so $\dim \mathcal{F}^2_{(x,y)} = 0$ for $(x,y) \neq (0,0)$. If $y \neq 0$ we have $\mathcal{F}^2_{(x,y)} = \mathcal{O}/J(f) = 0$. So f is regular for $y \neq 0$. If $y = 0$ and $x \neq 0$ we have $c(f_{(x,0)}) \le j(f_{(x,0)}) = 0$, and therefore the transversal type is S.

1) $\Rightarrow$ 2) For $y \neq 0$ f is regular so $\mathcal{F}^1_{(x,y)} = \mathcal{O}/J(f) = 0$. For $y = 0$ and $x \neq 0$ $f \sim \Pi S$, so $\mathcal{F}^1_{(x,y)} = 0$.

1) $\Leftrightarrow$ 4) Trivial. □

(1.15) Definition. Let $f \in I(S)$ be a line singularity with transversal type S, S as in Prop. (1.2), U a sufficiently small neighbourhood of 0 in $\mathbb{C}^{n+1}$ and $f_t = f + tg$ a deformation of f, with $g \in (y)^2$ and t sufficiently small.

1) We call f_t a S-Morsification if f_t has only codimension zero and codimension one singularities on $U \cap \{y = 0\}$ and only A_1 singularities outside $U \cap \{y = 0\}$.
2) We call a S-Morsification f_t an I(S)-Morsification if f_t is an I(S)-deformation (see (1.3)).

We now state and prove the Morsification theorem. Let $I = I(S)$ be as in Prop. (1.2). We list generators of I in the following way:

$I = (y_1^{k_1}, \dots, y_n^{k_n}, g_1, \dots, g_m)$ with $g_i \in \mathbb{C}[y_1, \dots, y_n]$ monomials.

Let $x = y_0$ for notational convenience.

Theorem. Let f be a line singularity with transversal type S, and assume that $f \in I(S)$ as in Prop. (1.2). Let

$F: \mathbb{C}^{n+1} \times T \longrightarrow \mathbb{C}$ be defined by

$$F(y,a,b,c) = f(y) + \sum_{j=1}^{n} a_j y_j^{k_j} + \sum_{j=0,i=1}^{n} b_{ji} y_j y_i^{k_i} + \sum_{j=1}^{m} c_j g_j + \sum_{(j,i)=(0,1)}^{(n,m)} d_{ji} y_j g_i$$

with $t = (a,b,c,d) \in \mathbb{C}^n \times \mathbb{C}^{n(n+1)} \times \mathbb{C}^m \times \mathbb{C}^{n \cdot m}$.

Then there exists an open neighourhood U of 0 in T, and an open dense subset $W \subset U$ such that for all $t \in W$

$f_t: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$

$f_t(y) = F(y,t)$

is an I(S)-Morsification of f. Moreover, the number of codimension one singularities is given by:

$$\# F_i S = h_i(f).$$

Proof: Consider the map Φ: $(\mathbb{C}^{n+1} \times T,0) \longrightarrow (\mathbb{C}^{n+1},0)$ defined by $\Phi = (\partial F/\partial y_0,\dots,\partial F/\partial y_n)$. One calculates that

$$\partial\Phi_i/\partial a_l = \delta_{il}k_i y_i^{k_i-1} \; ; \quad \partial\Phi_i/\partial b_{jl} = \delta_{il}k_l y_j y_i^{k_i-1} + \delta_{ij}y_l^{k_l}.$$

Therefore $\det(\partial\Phi/\partial b_{01} - y_0\partial\Phi/\partial a_1,\dots,\partial\Phi/\partial b_{n1} - y_n\partial\Phi/\partial a_1) = y_1^{k_1(n+1)}$.

If we set $K = (y_1^{k_1(n+1)},\dots,y_n^{k_n(n+1)})$, then $K \subset I_{n+1}(d\Phi)$, the ideal generated by the $(n+1) \times (n+1)$ minors of $d\Phi$. So $V(K)$ contains the singular locus of the map Φ, i.e. Φ is a submersion ouside $\Sigma \times T$. Therefore $\Phi^{-1}(0) - \Sigma \times T$ is either empty, in which case there are no critical points of f_t outside Σ for $t \in T$, or $\Phi^{-1}(0) - \Sigma \times T$ is smooth of dimension dim T. Then the projection

$$\pi\colon \Phi^{-1}(0) - \Sigma \times T \subset \mathbb{C}^{n+1} \times T \longrightarrow T$$

$$\pi(y,t) = t$$

is a local isomorphism above an open dense subset $W_0 \subset T$. Thus for $t \in W_0$ f_t has only A_1 singularities outside Σ.

Now we define

$$H_iF\colon \Sigma \times T \longrightarrow \mathbb{C}$$

$$H_iF(x,t) = H_if_t(x)$$

where the H_if_t are defined as in (1.13). Let

$$Z_i = \{ (x,t) \in \Sigma \times T\colon H_iF\,(x,t) = 0 \}.$$

The Z_i are hypersurfaces in $\Sigma \times T$, and it is left to the reader to check using the explicit description of F and the definition of H_iF that the Z_i are reduced. We now consider the natural projection

$p_i\colon Z_i \longrightarrow T$. This is a finite map because $h_i(f) < \infty$.

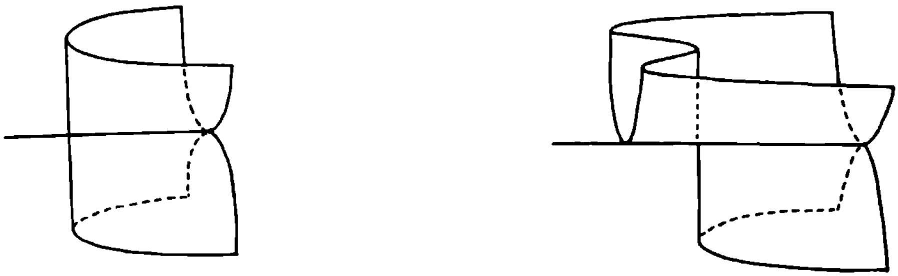
Consider $D_i := p_i(Z_i - U_i)$, where U_i is the subset in Z_i for which at $(x,0)$ the germ f_t is equivalent to F_iS. Because Z_i is reduced and the codimension one singularities F_iS are 2-determined in I(S) we have that $U_i \subset Z_i$ is open and dense. Let $W_i = T - D_i$. Then W_i is an open dense subset of T and the map:

$$p_i : Z_i - U_i \longrightarrow W_i$$

is an unramified covering of degree $h_i(f)$. Therefore $H_i f_t = 0$ intersects $\Sigma = (y = 0)$ transversally in $h_i(f)$ points for all $t \in W_i$, and those intersection points correspond to F_i points of f_t. To conclude the proof we take $W = (\bigcap_{i=1}^{s} W_i) \cap W_0$. □

(1.16) <u>Example.</u> $f = xz^2 + y^4 \in I(A_3)$; $h_1(f) = h_2(f) = 1$, $j(f) = 2$. We take the $I(A_3)$-Morsification $f_t = xz^2 + y^4 - ty^2z$.

$t = 0$: $t > 0$:



(1.17) It follows from the proof of Theorem (1.15) that the number of F_iS points in an I(S)-Morsification of $f \in I(S)$ is independent of the I(S)-Morsification. This will be used to give a S-Morsification, which is not an I(S)-Morsification.

<u>Example.</u> $f = x^2z^2 + y^4 + z^4 \in I(A_3)$. Then in an $I(A_3)$-Morsification $\# F_1A_3 = \# F_2A_3 = 2$; $\# A_1 = 6$. But in the A_3-Morsification $f_t = (xz + ty)^2 + y^4 + z^4$ we have that $\# F_1A_3 = 0$, $\# F_2A_3 = 4$ and $\# A_1 = 2$.

§2 The number of A_1 points in an I(S)-Morsification.

(2.1) In this paragraph we discuss the number of A_1 points in a S-Morsification. As example (1.17) shows, this number can very well depend on the S-Morsification. We therefore restrict ourselves to I(S)-Morsifications.

Theorem. Let f be a line singularity with transversal type A_1, A_2, A_3, D_4 or E_6, n = 2 if $S = E_6$. Let $f \in I(S)$ as in Prop. (1.2). Then:

$$j(f) < \infty, \text{ and } \# A_1 = j(f) - \Sigma\, j(F_i S) h_i(f)$$

where $\# A_1$ denotes the number of A_1 points in an I(S)-Morsification.

Remark. Although many calculations suggest that this formula should also be valid if $S = E_6$, $n > 2$, $S = E_7$ or E_8 if $n = 2$, our proof does not work in these cases, see (2.6).

(2.2) We give here a table in which the $j(F_i S)$ are listed:

S	F_1	F_2	F_3
A_1	1		
A_2	1	1	
A_3	1	1	
D_4	1	2	
E_6	1	2	2
E_7	1	3	
E_8	2	3	

Table 3

(2.3) Our proof heavily depends on a description of part of the Koszul homology on the partial derivatives of f. Let $R = \mathbb{C}\{x,y_1,\ldots,y_n\}$ and m be its maximal ideal. Let $f \in R$, such that $\dim_{\mathbb{C}}\Sigma(f) = 1$. Let $f_0 = \partial f/\partial x$, $f_i = \partial f/\partial y_i$; $i = 1,\ldots,n$, and $J(f) = (f_0,\ldots,f_n)$. Let $J(f) = q_0 \cap \ldots \cap q_r$ be a minimal primary decomposition of $J(f)$, with q_0 a m-primary component, if $J(f)$ has one, take $q_0 = R$ otherwise. Define $I(f) = q_1 \cap \ldots \cap q_r$. Then $I(f)$ is the smallest Cohen-Macaulay ideal with $I(f) \supset (f_0,\ldots,f_n)$. Because $\dim V(J(f)) = 1$, we have that $pd(R/I(f)) = n$.

Consider the following commutative diagram :

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & G_n & \longrightarrow \cdots \longrightarrow & G_1 & \longrightarrow & R & \longrightarrow & R/I(f) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow \varphi_n & & \uparrow \varphi_1 & & \| & & \uparrow & & \uparrow \\
0 & \longrightarrow & \Lambda^{n+1}F & \longrightarrow & \Lambda^n F & \longrightarrow \cdots \longrightarrow & \Lambda^1 F & \longrightarrow & R & \longrightarrow & R/J(f) & \longrightarrow & 0.
\end{array}$$

where:

1) The upper row is a projective resolution or R- modules of $R/I(f)$.

2) $F = R^{n+1}$, and the lower row is the Koszul complex on $\underline{f} = (f_0,\ldots,f_n)$.

3) Because G. is acyclic, and F is free, we get a map of complexes $\varphi_i : \Lambda^i F \longrightarrow G_i$, which extend the identity id: $R \longrightarrow R$.

Our starting point is the following

<u>Theorem.</u>[Pe, pg.173]:

$$\varphi_n^* : \mathrm{Ext}_R^n(R/I(f),R) \longrightarrow H^n(\Lambda^{\cdot}F^*,\underline{f}^*) \cong H_1(\underline{f},R)$$

is an isomorphism, where $H_1(\underline{f},R)$ is the the first Koszul homology group on the partial derivatives of f.

(2.4) Because we only need a part of the Koszul homology we define ideals $\tilde{I}(S)$. The proof of Theorem (2.1) will be slighty different for $S = A_1$, and we will not need an $\tilde{I}(A_1)$.

S	$\bar{I}(S)$
A_2	$(y_1,\ldots,y_n)$
A_3	$(y_1^2,y_2,\ldots,y_n)$
D_4	$(y_1,y_2)^2 + (y_3,\ldots,y_n)$
E_6	$(y^3,\ y^2z,\ z^2)$

Table 4

We write $\bar{I}$ for $\bar{I}(S)$ if no confusion is likely. Remark that $\bar{I}$ have been chosen in such a way that $\tilde{I} \supset J(f)$ for all $f \in I(S)$. If $S = E_6$, and $n > 2$, then the ideal should be taken *bigger*, namely $\bar{I} = \bar{I}(D_4)$. This is the reason that the proof does not work if $S = E_6$, and $n > 2$. Also, if $S = E_7$ or E_8, the ideals $\bar{I}$ that could have been defined are too big.

Consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & K_n \arrow[r] & \cdots \arrow[r] & K_1 \arrow[r] & R \arrow[r] & R/\bar{I} \arrow[r] & 0 \\
 & 0 \arrow[r] \arrow[u] & G_n \arrow[r] \arrow[u, "p_n"'] & \cdots \arrow[r] & G_1 \arrow[r] \arrow[u, "p_1"'] & R \arrow[r] \arrow[u, equal] & R/I(f) \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] & \Lambda^{n+1}F \arrow[r] \arrow[u] & \Lambda^{n}F \arrow[r] \arrow[u, "\varphi_n"'] & \cdots \arrow[r] & \Lambda^{1}F \arrow[r] \arrow[u, "\varphi_1"'] & R \arrow[r] \arrow[u, equal] & R/J(f) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

<u>Remarks:</u>

1) $K.$ is a projective resolution of $R/\bar{I}$ ($\mathrm{pd}(R/\bar{I}) = n$).

2) Because $I(f)$ was the smallest Cohen-Macaulay ideal such that $I(f) \supset J(f)$, and $\bar{I} \supset J(f)$, with $\bar{I}$ Cohen-Macaulay , we have that $I(f) \subset \bar{I}$. The maps p_i extend the identity id: $R \longrightarrow R$.

3) R is a local Cohen-Macaulay ring, so grade $(\bar{I}/I(f)) :=$

$\min\{i : \mathrm{Ext}^i_R(\bar{I}/I(f),R) \neq 0\} = \mathrm{codim}\ \mathrm{supp}(\bar{I}/I(f)) = n.$

(See [Ma, pg 103]).

By remark 3) we have that $\mathrm{Ext}^{n-1}_R(\bar{I}/I(f),R) = 0$. Therefore the map

$$\varphi := \varphi^*_n\, p^*_n : \mathrm{Ext}^n_R(R/\tilde{I},R) \longrightarrow \mathrm{Ext}^n_R(R/I(f),R) \cong H_1(\underline{f},R)$$

is injective.

(2.5) If we have a representative $(\alpha_0,\ldots,\alpha_n) \in R^{n+1}$ of a class in $H_1(\underline{f},R)$, we can associate to it a vector field

$$\alpha_0\partial/\partial x + \alpha_1\partial/\partial y_1 + \ldots + \alpha_n\partial/\partial y_n.$$

Contraction with the volume form $dx\wedge dy_1\wedge\ldots\wedge dy_n$ gives an identification of $H_1(\underline{f},R)$ with

$$H = \mathrm{Ker}\ (df\wedge : \Omega^n \longrightarrow \Omega^{n+1})\ \mathrm{modulo}(df\wedge\Omega^{n-1}).$$

<u>Definition.</u>

1) Let $\approx{H} \subset H$ be the submodule corresponding to $\mathrm{Im}(\varphi) \subset H_1(\underline{f},R)$. By abuse of notation we set $\varphi: \mathrm{Ext}^n_R(R/\bar{I},R) \longrightarrow H$.

2) $\bar{H} = \{$classes in H represented by $\alpha_0\hat{dx} + \alpha_1\hat{dy}_1 + \ldots + \alpha_n\hat{dy}_n$ with

$$\alpha_0 \in (y_1,\ldots,y_n)\}.$$

with $\hat{dy}_i = \iota_{\partial/\partial y_i}(dx\wedge dy_1\wedge\ldots\wedge dy_n)$ and $d\hat{x} = \iota_{\partial/\partial x}(dx\wedge dy_1\wedge\ldots\wedge dy_n)$, where ι denotes contraction.

<u>Lemma.</u> The same assumptions as in Theorem (2.1). Then:

1) $\tilde{H} \subset \iota_{T\bar{D}}\cdot\Omega^{n+1}$.

2) $\tilde{H} = \approx{H}$ if $S \neq A_1$

$\tilde{H} = 0$ if $S = A_1$.

(2.6) We give generators for $\mathrm{Im}(\Psi)$ and check Lemma (2.5) for the codimension zero singularity ΠS. The explicit description of the generators will be used in the proof of Theorem (2.1), see (2.8), and the check of Lemma (2.5) for ΠS will be essential in the proof of Lemma (2.5), see (2.7). It is left to the reader, using the explicit generators below, to check that $\approx{H} \subset \bar{H}$, and $\approx{H} \subset \iota_{T\bar{D}}\cdot\Omega^{n+1}$. Let $\{a_i\}$ ($i\geq 1$ if $S = A_2$ or A_3, $i\geq 0$ otherwise) be the generators of $\bar{I}$, ordered as in (2.4). Then $df = \Sigma\omega_i a_i$ with $\omega_i \in \Omega^1$.

A) $S = A_2$ or A_3. Then $\mathrm{Ext}^n_R(R/\bar{I},R)$ and thus $\mathrm{Im}(\Psi)$ have one generator. The generator of $\mathrm{Im}(\Psi)$ is $\omega_1\wedge\ldots\wedge\omega_n$. Now if

$f = 1/3\cdot y_1^3 + 1/2\cdot(y_2^2 + \ldots + y_n^2)$, resp.

$f = 1/4\cdot y_1^4 + 1/2\cdot(y_2^2 + \ldots + y_n^2)$, then one checks that

$\omega_1\wedge\ldots\wedge\omega_n = y_1\hat{d}x$, which is precisely the generator of $\bar{H}$.

B) $S = D_4$: $\mathrm{Im}(\Psi)$ has two generators namely

$$y_1(\omega_0\wedge\omega_1\wedge\hat{\omega}_2\wedge\ldots\wedge\omega_n) + y_2(\omega_0\wedge\hat{\omega}_1\wedge\omega_2\wedge\ldots\wedge\omega_n) \text{ and}$$
$$y_1(\omega_0\wedge\hat{\omega}_1\wedge\omega_2\wedge\ldots\wedge\omega_n) + y_2(\hat{\omega}_0\wedge\omega_1\wedge\omega_2\wedge\ldots\wedge\omega_n).$$

If $f = 1/3\cdot y_1^3 + 1/3\cdot y_2^3 + 1/2\cdot(y_3^2 + \ldots + y_n^2)$ then the generators are $y_1\hat{d}x$ and $y_2\hat{d}x$. Therefore $\approx{H} = \tilde{H}$ if $c(f) = 0$.

C) $S = E_6$: Again $\mathrm{Im}(\Psi)$ has two generators

$$y^2(\omega_0\wedge\omega_1) + z(\omega_0\wedge\omega_2) \text{ and } y(\omega_0\wedge\omega_2) + z(\omega_1\wedge\omega_2).$$

And for ΠE_6 i.e. $f = 1/4\cdot y^4 + 1/3\cdot z^3$ we get $z\hat{d}x$ and $y\hat{d}x$. So again $\approx{H} = \bar{H}$ if $c(f) = 0$.

D) Let us see why it goes wrong for example for $S = E_7$. Then the smallest ideal $\bar{I}$ such that $J(f) \subset \bar{I}$ for all $f \in I(S)$ is $(y^3, y^2z, z^2) = \bar{I}(E_6)$. Therefore we can use C) and calculate that if $f = y^3z + z^3$ we get the following generators for $\approx{H}$: $y^2\hat{d}x$ and $y\hat{d}x$. These do not generate $\bar{H}$ in this case.

(2.7) Let us now prove Lemma (2.5). We have that depth$(\bar{H}/\overset{\approx}{H}) = 0$, because it is supported at zero by the calculation of (2.6). We consider the following sequence:

$$0 \longrightarrow \bar{H}/\overset{\approx}{H} \longrightarrow H/\overset{\approx}{H} \longrightarrow H/\bar{H} \longrightarrow 0$$

By the depth lemma (see e.g. [E-G]), it suffices to prove that depth$(H/\overset{\approx}{H}) = 1$, because it follows that $\bar{H}/\overset{\approx}{H} = 0$.

We have the following identifications:

$$H = \mathrm{Ext}_R^n(R/I(f),R);$$

$$\overset{\approx}{H} = \mathrm{Ext}_R^n(R/\bar{I},R);$$

$$H/\overset{\approx}{H} = \mathrm{Ext}_R^n(\bar{I}/I(f),R).$$

The last identification comes from the fact that $\mathrm{Ext}_R^{n+1}(R/\bar{I},R) = \mathrm{Ext}_R^{n-1}(\bar{I}/I(f),R) = 0$. Dualizing the upper two rows of the diagram in (2.4) we get:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & & \\
0 & \mathrm{Ext}_R^n(R/\bar{I},R) \arrow[l] \arrow[d] & K_n^* \arrow[l] \arrow[d] & \cdots \arrow[l] & K_1^* \arrow[l] \arrow[d] & R^* \arrow[l] \arrow[d, equal] & 0 \arrow[l] \arrow[d, equal] \\
0 \arrow[u, equal] & \mathrm{Ext}_R^n(R/I(f),R) \arrow[l] \arrow[d] & G_n^* \arrow[l] & \cdots \arrow[l] & G_1^* \arrow[l] & R^* \arrow[l] & 0 \arrow[l] \\
 & \mathrm{Ext}_R^n(\bar{I}/I(f),R) & & & & &
\end{tikzcd}$$

The rows are exact because grade$(I(f))$ = codim $V(I(f)) = n$, as well grade $(\bar{I}) = n$. Thus

$$0 \longleftarrow \mathrm{Ext}_R^n(\bar{I}/I(f),R) \longleftarrow G_n^* \longleftarrow G_{n-1}^* \oplus K_n^* \longleftarrow \cdots \longleftarrow K_1^* \longleftarrow 0$$

is exact, proving that pd$(H/\overset{\approx}{H}) = n$, and thus depth $(H/\overset{\approx}{H}) = 1$.

For $S = A_1$, we have for πS that $\bar{H} = 0$. This is true since H is generated by $\hat{d}x$, and $R/I \cong \mathrm{Ext}_R^n(R/I(f),R) \cong H$. Therefore the depth lemma applied to the sequence

$$0 \longrightarrow \bar{H} \longrightarrow H \longrightarrow H/\bar{H} \longrightarrow 0$$

gives the desired result. □

(2.8) We are now ready to prove Theorem (2.1). Let $f \in I(S)$, $j(f) < \infty$, with the same restrictions on S as in (2.1). Let $f + tg : (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a deformation with $g \in I(S)$. We define

$$\mathcal{M} = I/T\bar{\mathcal{D}}(f+tg),$$

which we identify with

$$\mathcal{M} = I \cdot \Omega^{n+1}/(df+tdg)\wedge\iota_{T\bar{\mathcal{D}}}\cdot\Omega^{n+1},$$

where the differentiation is not taken with respect to the variable t. Let

$$F: (\mathbb{C}^{n+1}\times \mathbb{C},0) \xrightarrow{(f+tg,t)} (\mathbb{C}\times\mathbb{C},0) \xrightarrow{(0,t)} (\mathbb{C},0).$$

If t is not a zero divisor of $\mathcal{M}$, then $F_*\mathcal{M}$ is a flat $\mathbb{C}\{t\}$ module, hence locally free, since $(\mathbb{C},0)$ is regular. The theorem then easily follows, because at an A_1 point q of $f+tg$: $\mathbb{C}^{n+1}\times t \longrightarrow \mathbb{C}$ we have $\dim_{\mathbb{C}}(\mathcal{M}_q/F^*(m_t)\cdot\mathcal{M}_q = 1$, where m_t is the maximal ideal of $\mathcal{O}_{\mathbb{C},t}$. See also (1.14). It remains to show that t is not a zero divisor.

A) $S = A_1$: Assume:

$t\omega = (df+tdg)\wedge\eta$; $\eta \in \iota_{T\bar{\mathcal{D}}}\cdot\Omega^{n+1}$ and write $\eta = B + t\cdot A$.

then: $\omega = (df+tdg)\wedge A + dg\wedge B$, $df\wedge B = 0$, and $\text{class}(B) \in \bar{H} \subset H$.

By Lemma (2.5) we have that $B = df\wedge\theta$ for a certain θ and thus $\omega = (df+tdg)\wedge(A - dg\wedge\theta)$, proving that t is not a zero divisor.

B) $S = A_2, A_3, D_4, E_6$: We only consider $S = A_2$ and A_3, leaving the remaining cases to the reader. Let $\{a_i\}$ $i\geq 1$ be the generators of $\bar{I}$. Then: $df = \Sigma\, \omega_i a_i$; $dg = \Sigma\, \rho_i a_i$; $\omega_i,\rho_i \in \Omega^1$. Define $\Omega := \omega_1\wedge\ldots\wedge\omega_n$; $\Xi := (\omega_1+ t\rho_1)\wedge\ldots\wedge(\omega_n+ t\rho_n)$.

Remark that:

1) $df\wedge\Omega = 0$; $(df + tdg) \wedge \Xi = 0$.

2) If we write $\Xi = \Omega + t\,\xi$ then it follows that

$dg\wedge\Omega + (df+tdg)\wedge\xi = 0$.

Now assume: $t\omega = (df+tdg)\wedge\eta$, $\eta \in \iota_{\bar{T}D}\cdot\Omega^{n+1}$ and write $\eta = B + t\cdot A$. Then: $\omega = (df+tdg)\wedge A + dg\wedge B$, $df\wedge B = 0$, and class $(B) \in \tilde{H}$ by Lemma (2.5) 1). So by Lemma (2.5) 2) and (2.6) A) we have $B = Q\cdot\Omega + df\wedge\theta$ for certain Q and θ. Using 2):

$\omega = (df + tdg)\wedge(A - Q\cdot\xi - dg\wedge\theta)$, showing that t is not a zero divisor.

§3. The topology of the Milnor fibre.

(3.1) In this last paragraph we determine the homotopy type of the Milnor fibre. Following Siersma closely ([Si 2]), this will be done by a generic deformation, having only codimension ≤ 1 and A_1 singularities in the deformation. Of course, a good knowledge of the Milnor fibres of these elementary singularities is necessary. We first compute the homology groups of the Milnor fibre (all coefficients are in $\mathbb{Z}$), and with additional information about the fundamental group, the homotopy type can be determined, cf. ([Si 2 Prop. 6.1]).

We state the theorem:

Main Theorem. Let $f: (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a line singularity as in Prop. (1.2), not of type ΠS. Then the Milnor fibre G of f is homotopy equivalent to $\bigvee_{\epsilon} S^{n-1} \bigvee_{\mu} S^{n}$, with $\mu = \Sigma\, \alpha_i h_i - \mu(S) + \#\, A_1$, where h_i denotes the number of F_iS points and $\#\, A_1$ the number of A_1 points in an I(S)-Morsification, and:

S	α_1	α_2	α_3	ϵ
A_1	2			0
A_2	3	4		0
A_3	2	4		$\begin{cases} 0 \text{ if } h_2 \neq 0 \\ 1 \text{ if } h_2 = 0 \end{cases}$
D_4	3	8		≤ 1; 0 if $h_2 \neq 0$
E_6	4	8	9	$\begin{cases} 0 \text{ if } h_2 + h_3 \neq 0 \\ 2 \text{ if } h_2 + h_3 = 0 \end{cases}$
E_7	6	14		$\begin{cases} 0 \text{ if } h_2 \neq 0 \\ 1 \text{ if } h_2 = 0 \end{cases}$
E_8	10	12		0

table 5

(3.2) Let $f: (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a line singularity as in Prop. (1.2). Let ϵ_0 be an admissible radius for the Milnor fibration (see [Mi]), i.e. such that for all ϵ with $0 < \epsilon \leq \epsilon_0$ $f^{-1}(0) \pitchfork \partial B_\epsilon$, as a stratified set. For each admissible $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ such that $f^{-1}(s) \pitchfork \partial B_\epsilon$ for all
$0 \leq s \leq \delta(\epsilon)$. We now fix $\epsilon \leq \epsilon_0$ and consider $\delta \leq \delta(\epsilon)$ and take the representative

$$f: X_\Delta := f^{-1}(\Delta) \cap B_\epsilon \longrightarrow \Delta, \text{ where } \Delta \text{ is the disc of radius } \delta.$$

Lemma. Let f be as above, and f_t be a deformation (locally trivial outside the origin). Consider the restriction:

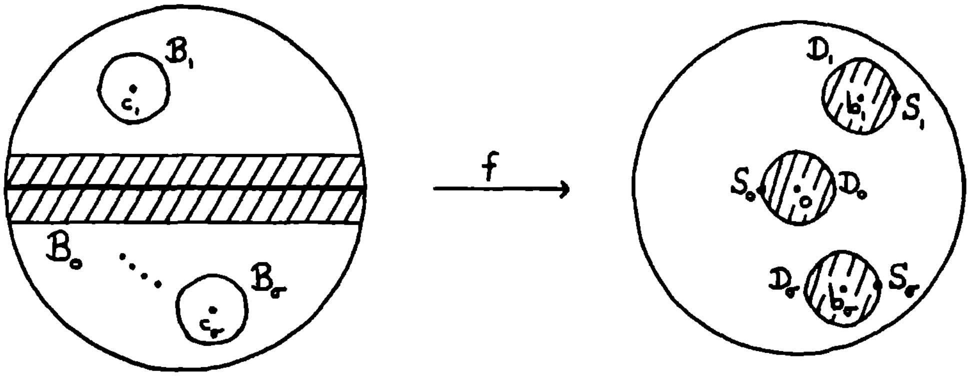
$$f_t: X_{\Delta,t} := f_t^{-1}(\Delta) \cap B_\epsilon \longrightarrow \Delta.$$

For t, $\delta > 0$ sufficiently small, we have

1) $f_t^{-1}(s) \pitchfork \partial B_\epsilon$ for all $s \in \Delta$.
2) Above the boundary circles $\partial\Delta$ the fibrations induced by f and f_t are equivalent.
3) X_Δ and $X_{\Delta,t}$ are homeomorphic.

For the proof we refer to [Si 2]. Although he states it only for $S = A_1$, the proof goes on in our case, without difficulties.

(3.3) Let f_t be a S-Morsification of f. We suppose that f: $X_\Delta \longrightarrow \Delta$ satisfies the conditions of Lemma (3.2). By abuse of notation we again denote f_t by f. Let $c_1,\dots,c_\sigma$ be the A_1 points of f, with critical values $b_1,\dots,b_\sigma$. Let the critical value zero correspond to the non isolated singularities of f. We can assume that all critical values are distinct.



Choose:

1) Small disjoint balls B_i around c_i ($i = 1,\dots,\sigma$) and a small tube B_0 around Σ.

2) Small disjoint discs D_i around b_i ($i = 1,\dots,\sigma$) and D_0 around 0 such that $f^{-1}(s) \pitchfork \partial B_i$ for all $s \in D_i$.

3) Points $s_i \in \partial D_i$, and a point $s \in \partial \Delta$.

We introduce the following notation:

$B = B_\epsilon$	$\Sigma = \Sigma \cap B_0$
$E^i = B_i \cap f^{-1}(D_i)$	$G^i = B_i \cap f^{-1}(s_i)$
$E = B \cap f^{-1}(\Delta)$	$G = B \cap f^{-1}(s)$

The following proposition is proved in [Si 2, (2.8)]

<u>Proposition.</u> $H_*(E,G) = \bigoplus_{i=0}^{\sigma} H_*(E^i, G^i)$.

(3.4) At the points c_i, we have an A_1 singularity. At these points, the topology of the G^i is well-known (see e.g. [Mi]); G^i is homotopy equivalent to S^n. Therefore

$$H_{n+1}(E,G) = H_{n+1}(E^0,G^0) \oplus \mathbb{Z}^\sigma$$

$$H_k(E,G) = H_k(E^0,G^0) \text{ if } k \neq n+1.$$

For later use we also note the following:

<u>Lemma.</u> $\pi_1(G)$ is a factor group of $\pi_1(G^0)$.

<u>Proof:</u> This is an easy consequence of Van Kampen's Theorem, together with $\pi_1(S^n) = 0$ if $n \geq 2$. □

(3.5) We now concentrate on (E^0,G^0). We make this pair into a fibre space with basis Σ, locally trivial outside the special points. Therefore we consider:

$$\pi: B_0 \longrightarrow \Sigma$$
$$(x,y) \longrightarrow x, \text{ and}$$

$$\Phi: B_0 \longrightarrow \mathbb{C} \times \Sigma$$
$$(x,y) \longrightarrow (f(x,y),x),$$

and we choose small disjoint discs in Σ around each special point (i.e. the F_iS points). Call these discs $W_1,\dots,W_\tau$. We set:

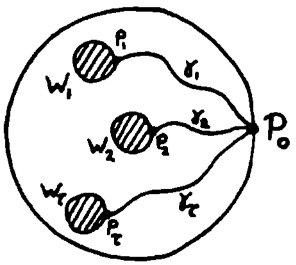
$W = W_1 \cup \dots \cup W_\tau$, $M = \overline{\Sigma - W}$, $E_Y = \pi^{-1}(Y) \cap E^0$ and $G_Y = E_Y \cap G^0$ if $Y \subset \Sigma$.

<u>Proposition.</u> For E^0 sufficiently small we have that

π: $(E_M, G_M) \longrightarrow M$ is a locally trivial fibre bundle with fibres equivalent to the fibre pair $(\bar{E},\bar{G})$ of the isolated singularity S.

<u>Proof:</u> This is Siersma's Proposition (4.7) ([Si 2]). To apply it we only have to check that the critical locus of Φ is equal to M. This critical locus is given by $\partial f/\partial y_1 = \dots = \partial f/\partial y_n = 0$. By Lemma I(9.9) of Pellikaan ([Pe]) we have that $\partial f/\partial x \in (\partial f/\partial y_1,\dots,\partial f/\partial y_n)$ along M. So the critical locus of Φ is equal to the critical locus of f along M, which by assumption is M. □

(3.6) We choose a point $p_0 \in \partial\Sigma$, points $p_i \in \partial W_i$, and a system of nonintersecting paths $\gamma_1,\dots,\gamma_\tau$ from p_0 to p_i in the usual way (see the figure). Let $C = \bigcup_{i=1}^{\tau} \gamma_i$.



Applying the relative Mayer-Vietoris sequence ([Do pg. 51]) to the excisive triad $(G_{W\cup C};\ G_W, G_C) \subset (E_{W\cup C}; E_W, E_C)$ we get the following long exact sequence:

$$H_*(E_{W\cup C}, G_{W\cup C}) \xrightarrow{d_*[-1]} H_*(E_{W\cap C}, G_{W\cap C})$$

$$H_*(E_W, G_W) \oplus H_*(E_C, G_C)$$

Remarks:

1) $H_*(E_{W\cup C}, G_{W\cup C}) = H_*(E^0, G^0)$, because $W \cup C$ is homotopy equivalent to Σ, and the homotopy lifting property of π: $(E_M, G_M) \longrightarrow M$, see (3.5).

2) $H_k(E_{W\cap C}, G_{W\cap C}) = \bigoplus_{i=1}^{\tau} H_k(E_{p_i}, G_{p_i}) = \bigoplus_{i=1}^{\tau} H_{k-1}(G_{p_i}) =$
$\begin{cases} Z^{\tau\cdot\mu(S)} & \text{if } k = n \\ 0 & \text{if } k \neq n, \end{cases}$
because at each p_i we have a transversal S singularity with Milnor number $\mu(S)$.

3) $H_k(E_C, G_C) = H_k(\bar{E}, \bar{G}) = \begin{cases} \mu(S) & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$

4) Obviously it is important to study $H_k(E_W, G_W) = \bigoplus_{i=1}^{\tau} H_k(E_{W_i}, G_{W_i})$, i.e. the Milnor fibres of the codimension one singularities.

(3.7) <u>Lemma.</u> Theorem (3.1) is true for the codimension one singularities.

<u>Proof:</u> We first remark that, because all codimension one singularities are quasihomogeneous (see table 2), we can take $\epsilon = \delta = \infty$ in the Milnor construction. The second remark is, that if

$f = g(x,y) + z_1^{q_k} \ldots + z_k^{q_k}$ then the Milnor fibre of f is obtained by taking the join of the Milnor fibre of g with q_1 points, then q_2 points, etc. (up to homotopy type). Now, if $f = x \cdot y^k$, then the Milnor fibre is given by $x \cdot y^k = 1$, and thus by $x = 1/y^k$, because $y = 0$ does not intersect the Milnor fibre. Therefore the Milnor fibre is homotopy equivalent to the graph of a function, i.e. to its domain which is $\mathbb{C} - 0$. This implies the result for F_1A_1, F_1A_2, F_2A_2, and F_2A_3. It also gives the result for F_2D_4, F_2E_6, F_3E_6, F_1E_8 and F_2E_8 because the functions $xw^2 + y^3 + z^3$, $xy^4 + z^3$, $xz^3 + y^4$, $xy^5 + z^3$ and $xz^3 + y^5$ deform into the above mentioned functions (in this order!) without generating other codimension one singularities or A_1 points. This is checked by a little computation. The assertion therefore follows from Lemma (3.2). For F_1A_3, F_1D_4, F_1E_6 and F_1E_7 we use

$xz^k + y^l z = 1$, $x = 1 - y^l z/z^k$, because $z = 0$ does not intersect the Milnor fibre. So in this case the Milnor fibre is homotopy equivalent to $\mathbb{C}^2 - \{z = 0\}$, i.e. S^1. The last one is F_2E_7. The Milnor fibre is the union of:

A: $x = (1 - z^3 - y^5)/y^3 z$, $yz \neq 0$

B: $z = 0$ and $y^5 = 1$

C: $y = 0$ and $z^3 = 1$

Therefore, we have to attach eight cells to A, which is homotopy equivalent to $\mathbb{C}^* \times \mathbb{C}^*$ (graph of a function). Take for example the complex line $z = 1$, $y = 0$. $\{z = 1\} \cap A = \{ x = -y^2, y \neq 0\}$, so the complex line fills up the hole of $\mathbb{C}^* \times \{1\}$. If we take into account all eight complex lines, we get the desired result.□

(3.8) Lemma. Let $f: (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a line singularity. Consider the Milnor fibre G of f and the Milnor fibre G' of $f_c(y) = f(c,y)$ for $c \neq 0$ sufficiently small. Then $G' \subset G$ and the map induced by this inclusion $H_{n-1}(G') \longrightarrow H_{n-1}(G)$ is surjective.

Proof: The fact that $G' \subset G$ is trivial. By the long exact sequence of the pair (G, G') it suffices to show that $H_{n-1}(G,G') = 0$. We use Milnor's original construction of the Milnor fibre (see [Mi]). Therefore let $\epsilon > 0$ sufficiently small, $S_\epsilon = \{(x,y): \|(x,y)\| \leq \epsilon\}$, and $K = S_\epsilon \cap f^{-1}(0)$. The singular locus of K is equal to $S_\epsilon \cap \Sigma$, which is a circle in S_ϵ. We can assume that $c \in S_\epsilon \cap \Sigma$. Take a small open tube S along $S_\epsilon \cap \Sigma$, seen as a disc bundle over $S_\epsilon \cap \Sigma$. Let B be the fibre above c. Now let G be a Milnor fibre of f. Consider $G-S$, which is homotopy equivalent to G. Now $\overline{(G-S) \cup B}$ has the same homotopy type as $S_\epsilon - \overline{((G-S) \cup B}$. This is because $S_\epsilon - \overline{((G-S) \cup B}$ is homotopy equivalent to $S_\epsilon - \overline{((G-S) \cup B'}$, where B' is a fibre of the disc bundle over a point $c' \neq c$ (Use the open book structure along $S_\epsilon \cap \Sigma$). Now, by Milnor's fibration Theorem $S_\epsilon - \overline{((G-S) \cup B'}$ has the same homotopy type as $\overline{(G-S) \cup B}$. By Alexander duality, as in Milnor ([Mi 6.2]), we get $H_{n-1}(G \cup B) = 0$. But because (after a diffeomorphism) B can be viewed as a Milnor ball for the transversal singularity, we get $H_{n-1}(G,G') = 0$. □

Remark. The assumption that f is a line singularity can be weakened to the condition that f has an irreducible critical locus.

(3.9) We take the Mayer-Vietoris sequence of (3.6) and fill in the homology groups. We assume that the number of F_iS points is h_i and we put $\tau = \Sigma h_i$. Denote by b_i resp. a_i the rank of the n-th resp. (n-1)-st homology group of the Milnor fibre of F_iS. We get:

$H_k(E^0,G^0) = 0$ if $k \neq n$, $n+1$ and the following exact sequence:

$$0 \longrightarrow \mathbf{Z}^{\Sigma b_i h_i} \longrightarrow H_{n+1}(E^0,G^0) \longrightarrow \mathbf{Z}^{\tau\mu(S)} \longrightarrow \mathbf{Z}^{\Sigma a_i h_i} \oplus \mathbf{Z}^{\mu(S)} \longrightarrow H_n(E^0,G^0) \longrightarrow 0$$

We distinguish between two cases:

1) There is an F_iS point with $b_i \neq 0$ and $a_i = 0$. Then $H_n(E^0,G^0) = 0$. In fact, take a representative η of a class in $H_n(E^0,G^0)$, and a $\eta' \in H_n(E_W,G_W) \oplus H_n(E_C,G_C) = \mathbf{Z}^{\Sigma a_i h_i} \oplus \mathbf{Z}^{\mu(S)}$, which maps to η. By the local surjectivity (Lemma (3.8)), we can transport this cycle η' along C to the F_iS point with $b_i \neq 0$ and $a_i = 0$. But then it is clear that η represents the zero class in $H_n(E^0,G^0)$.

2) There is no such point. In the cases $S = A_3$, D_4 and E_7, we can conclude that $H_n(E^0,G^0)$ is a cyclic group. In fact, in these cases we have that at a point F_iS $H_n(E_{W_i},G_{p_i}) \cong \mathbf{Z}$, and all cycles which are not zero in $H_n(E^0,G^0)$ can be transported to this F_iS point as in 1). With the same argument we have that if $S = E_6$, and there are no F_2E_6 and F_3E_6 points, that $H_n(E^0,G^0)$ is a factor group of $\mathbf{Z} \oplus \mathbf{Z}$.

(3.10) To investigate case 2) of (3.9) further, we need ad hoc arguments. First we write down the results. We now restrict ourselves to I(S)-Morsifications.

Proposition. Let f be a line singularity with transversal type S as in Prop. (1.2), not of type πS. Then the homology groups of the Milnor fibre G of f are given by:

$H_n(G) = \mathbb{Z}^{\mu+\epsilon}$; $H_{n-1}(G) = \mathbb{Z}^{\epsilon}$ with $\mu = \Sigma\, \alpha_i h_i - \mu(S) + \# A_1$, and with the same notations as in (3.1).

For the proof (see (3.13)), we use two lemmas and the following result of Van Straten.

Theorem. ([St Theorem 4.4.12]) Let f: $(\mathbb{C}^3,0) \longrightarrow (\mathbb{C},0)$ be a germ of a holomorphic function, with no multiple factors. Let G be the Milnor fibre of f. Then

$b_1(G) \geq \#\,\{$ irreducible components of $f = 0\} - 1$.

(3.11) **Lemma.** Let f: $(\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a line singularity with transversal type S.

If $S = A_3$ or E_7 and $h_2(f) = 0$ then

$f \sim g(x,y_1,y_2) + y_3^2 + \ldots + y_n^2$ with g reducible.

If $S = E_6$ and $h_2(f) + h_3(f) = 0$ then

$f \sim g(x,y_1,y_3) + y_2^3 + y_4^2 + \ldots + y_n^2$, where g: $(\mathbb{C}^3,0) \longrightarrow (\mathbb{C},0)$ is a line singularity with transversal type A_3 and g reducible.

Proof: We give the proof in the $S = A_3$ case, leaving the remaining cases to the reader. Modulo functions of order > 1 we can assume (see (1.2)):

$$f = Ay_1 + \sum_{i=2}^{n} 2\cdot B_i y_1^2 y_i + \sum_{i=2}^{n} x^{\alpha_i} y_i;\ A,\ B_i \in \mathbb{C}\{x\}.$$

We may suppose: $\alpha_i \geq \alpha_{i+1}$ for all $i \geq 2$. But then it follows, because $h_2(f) = 0$ that $\alpha_i = 0$ for $i \geq 3$. By the change of coordinates:

$y_i \longrightarrow y_i - B_i y_1^2$; $i \geq 3$, we get that modulo stable equivalence:

$f \sim g := A'y_1^4 + 2B_2y_1^2y_2 + x^{\alpha_2}y_2^2$. Again because $h_2(f) = 0$ we get that $B_2^2 - A'x^{\alpha_2}$ is a unit in $\mathbb{C}\{x\}$. Therefore either A' is a unit and $\alpha_2 = 0$ in case we have $f \sim \Pi A_3$, or B_2 is a unit. Making succesively the change of coordinates $y_2 \longrightarrow y_2 - (A'/2B_2)y_1^2$ and using the finite determinacy theorem (1.7), we can assume that modulo functions of order > 1 $g \sim 2B_2y_1^2y_2 + x^{\alpha_2}y_2^2$. Now the higher order terms which can destroy the reducibility are of the form Qy_1^k with $Q \in \mathbb{C}\{x\}$. We then make the change of coordinates $y_2 \longrightarrow y_2 - (Q/2B_2)y_1^{k-2}$. Using the finite determinacy theorem, we conclude that g is reducible. □

(3.12) <u>Lemma.</u> Let $f: (\mathbb{C}^3,0) \longrightarrow (\mathbb{C},0)$ be a line singularity with transversal type D_4. If f is not adjacent to $xy^3 + z^3$ then f is reducible.

<u>Proof:</u> Modulo $(y,z)^4$ we can write

$$f = Ay^3 + By^2z + Cyz^2 + Dz^3;\ A,\ B,\ C \text{ and } D \in \mathbb{C}\{x\}.$$

We can assume that not both A and D are units, because otherwise $f \sim \Pi D_4$. Let $\mathrm{order}(A) \geq \mathrm{order}(D)$. We write down the following deformation: $f_t = (A + tx)y^3 + By^2z + Cyz^2 + (D + t)z^3$. After a change of coordinates: $f_t = xy^3 + B'y^2z + C'yz^2 + z^3$ for generic t, and $\mathrm{order}(B') = \mathrm{order}(B)$, $\mathrm{order}(C') = \mathrm{order}(C)$. By assumption we have that f_t is not right equivalent to $xy^3 + z^3$. Therefore B' or C' is a unit, and thus B or C is a unit. Let us without loss of generality B is a unit. But then we can perform the change of coordinates $z \longrightarrow z - 1/3(A/B)y$, and we can argue as in the proof of Lemma (3.11) to conclude that f is reducible. □

(3.13) Now we can give the proof of Prop. (3.10). We first take the cases $S = A_3$ or E_7 and $h_2(f) = 0$. Then Lemma (3.11) says that $f = g(x,y_1,y_2) + y_3^2 + \ldots + y_n^2$, with g reducible. The Milnor fibre G of f is a repeated suspension of the Milnor G'' of g. But $b_1(G'') \geq 1$ by Van Straten's Theorem (3.10) and thus $b_{n-1}(G) \geq 1$. We have seen that $H_{n-1}(G)$ is a cyclic group (see (3.9), and therefore $H_{n-1}(G) = \mathbf{Z}$. For $S = E_6$ and $h_2(f) + h_3(f) = 0$, we use the description of f as in Lemma (3.11). Therefore the Milnor fibre G has the homotopy type of the suspension of 3 points with the Milnor fibre of a line singularity with transversal type A_3 and $h_2(f) = 0$. Thus $b_{n-1}(G) = 2$ and $H_{n-1}(G) = \mathbf{Z} \oplus \mathbf{Z}$. For $S = D_4$ and $h_2(f) = 0$, we can assume: $f = g(x,y_1,y_2) + y_3^2 + \ldots + y_n^2$. If g is reducible it follows that $H_{n-1}(G) = \mathbf{Z}$ as in the above cases. Now if g is irreducible, then we can write down a deformation in which $xy^3 + z^3$ occurs. For this deformation we can make the whole construction (3.2) --- (3.6), (3.8) and (3.9). Because the Milnor fibre of $xy^3 + z^3$ is homotopy equivalent to $S^2 \vee S^2$ (see the proof of (3.7)), we can use case 1) of (3.9) to conclude that $H_{n-1}(G) = 0$. □

(3.14) Next we need additional information about the fundamental group, in order to apply Prop. (6.1) of ([Si 2]).

<u>Proposition.</u> Let $f: (\mathbf{C}^{n+1},0) \longrightarrow (\mathbf{C},0)$ be a line singularity with transversal type S as in Prop. (1.2), and G be the Milnor fibre of f. Assume that f is not right equivalent to πS. Then $\pi_1(G)$ is a cyclic group. In particular $\pi_1(G) = H_1(G)$.

Proof: If $n \geq 3$ we have by Kato and Matsumoto ([K-M]) that $\pi_1(G) = 0$. Suppose $n = 2$. We take a S-Morsification of f. By Lemma (3.2), it suffices to show that $\pi_1(G^0)$ is a cyclic group. We proceed by induction on the number of codimension one singularities τ in the deformation. For $\tau = 1$, it follows from Lemma (3.7). (Remark that F_1E_6 can not occur if $n = 2$). We apply Van Kampen's Theorem to

$E^1 = E_{\gamma_1 \cup \ldots \cup \gamma_{\tau-1} \cup W \cup \ldots \cup W_{\tau-1}}$ and $E^2 = E_{\gamma_\tau \cup W_\tau}$. Then $E^1 \cup E^2$ is homotopy equivalent to G^0 and $E^1 \cap E^2 = E_{P_0}$. Then we have the pushout diagram:

$$\begin{array}{ccc} \pi_1(E_{P_0}) & \twoheadrightarrow & \pi_1(E^2) \\ \downarrow & & \downarrow \\ \pi_1(E^1) & \longrightarrow & \pi_1(G^0) \end{array}$$

By induction we have that $\pi_1(E^1)$ and $\pi_1(E^2)$ are cyclic groups. The indicated surjectivity follows from Lemma (3.8), and $\pi_1(E^2) = H_1(E^2)$. Then it follows that $\pi_1(G^0)$ is a cyclic group. □

(3.15) Proof of theorem (3.1):

For all cases, except $S = E_6$ and $h_2(f) + h_3(f) = 0$, we have enough information to conclude the theorem. We refer to Siersma ([Si 2 Prop. 6.1, Theorem 6.3]) for the proof of the homotopy equivalence. For $S = E_6$ and $h_2(f) + h_3(f) = 0$. we use Lemma (3.11) to conclude that the Milnor fibre G of f has the homotopy type of the join of three points with the Milnor fibre of a line singularity with transversal type A_3 and $h_2(f) = 0$. □

(3.16) The above result raises the following question.

Question. Is it always true that for a line singularity (or more general a function f with an irreducible curve as critical locus) that the Milnor fibre is homotopy equivalent to $\bigvee_{\alpha} S^{n-1} \bigvee_{\beta} S^{n}$ for certain α and β?

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CHAPTER TWO

THE VIRTUAL NUMBER OF D_∞ POINTS

§ 1 Introduction

(1.1) In this paper we study germs of holomorphic functions $f: (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ with a one dimensional critical locus $\Sigma(f) = \Sigma$ and transverse type A_1. The last condition means that around a general point $p \in \Sigma \setminus \{0\}$ the function f is right equivalent to the A_∞ singularity, i.e. locally $f(x_0,\dots,x_n) = \sum_{i=1}^{n} x_i^2$ in suitable local coordinates. Throughout the paper, we give Σ the reduced structure, defined by the ideal $I = \mathcal{I}(\Sigma)$. These singularities have been studied by Pellikaan [Pe 1], [Pe 2] and Siersma [Si 1], [Si 2]. We recall some of the most important ideas and results, in order to explain the title of the paper.

(1.2) We denote the above situation by $(f,\Sigma,0)$.

Definition: [Pe 1 I 7.4]

A deformation $(F,X,0)$ *of* $(f,\Sigma,0)$ *over an analytic base* $(S,0)$ *is given by:*

i) A flat embedded deformation $g: (X,0) \longrightarrow (S,0)$ *of* $(\Sigma,0)$. *We denote by* $\tilde{I}$ *the ideal defining* $(X,0)$ *in* $(\mathbb{C}^{n+1} \times S, 0)$.

ii) A function $F: (\mathbb{C}^{n+1}\times S,0) \longrightarrow (\mathbb{C},0)$ *such that* $(\partial_0 F,\dots,\partial_n F,F) \subseteq \tilde{I}$ *and* $F(-,0) = f(-): (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$.

We will write $f_s(-): (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ for $F(-,s)$ and similarly Σ_s for $g^{-1}(s)$.

(1.3) **Definition:**

A Morsification $(F,X,0)$ of $(f,\Sigma,0)$ *is a deformation of* $(f,\Sigma,0)$ *such that for generic* $s \in S$:

i) Σ_s *is smooth.*

ii) f_s *has only* A_∞ *and* D_∞ *singularities on* Σ_s *(this of course* *implies i)).*

iii) f_s *has only* A_1 *(Morse) singularities outside* Σ_s.

(D_∞ is the right equivalence class of $f = x_0x_1^2 + x_2^2 + \ldots + x_n^2$.)

This should be seen as an analogue of the Morsification in the isolated singularity case (cf. [Br]). Because of the restricted class of deformations which are allowed, one can build up the Milnor fibre M of the germ of the function M out of the Milnor fibres of the singularities mentioned in ii) and iii) of the definition. Siersma [Si 2] treated the case that $(\Sigma,0)$ is a one dimensional isolated complete intersection singularity, using a result of Pellikaan which states that in this case there always exists a Morsification of $(f,\Sigma,0)$. [Pe 1 I 7.18]. The result of Siersma is the following:

Theorem: [Si 2] *If* $(f,\Sigma,0)$ *is as above,* $(\Sigma,0)$ *an isolated complete intersection singularity, then the Milnor fibre* M *of* f *is homotopy equivalent to a wedge of* μ n - *spheres if the number of* D_∞ *points in the Morsification is greater than zero, if this number is equal to zero then* M *is homotopy equivalent to a wedge of* $\mu+1$ n - *spheres wedged with a* n -1 *sphere, where* $\mu = \# A_1 + \# D_\infty + \mu(\Sigma) -1$, $\#A_1$ *and* $\# D_\infty$ *denoting the number of them occuring in a Morsification. (These numbers are independent of the Morsification* [Pe 1] I 7.18].) *Moreover* $\mu(\Sigma)$ *is the Milnor number of* Σ*, as defined by Buchweitz and Greuel* [B-G].

(1.4) Although Siersma states this theorem only for the case that Σ is a complete intersection, his method works whenever a Morsification of $(f,\Sigma,0)$ exists, except that in case the number of D_∞ points is equal to zero one can only conclude that the (n-1)st Betti number of the Milnor fibre is smaller or equal to one. In general one can not expect that a Morsification exists, because a necessary condition is that the singular locus is smoothable. It is well-known that non smoothable curve singularities exist. We refer the reader to [Mu], [Pi] and [Gr 2]. But even if Σ is smoothable, a Morsification need not to be found.

A simple example is given by $f = xyz$ (see 3.1 and below).
Another problem is to give algebraic formulas for the number of A_1 and the number of D_∞ points in a Morsification, if a Morsification exists. For the case that the singular locus is a complete intersection these formulas were given by Pellikaan.

Let $J(f) = (\partial_0 f, \ldots, \partial_n f)$ be the Jacobi ideal, and define $j(f) = \dim_{\mathbb{C}}(I/J(f))$. One easily calculates that $j(A_\infty) = 0$, $j(D_\infty) = 1$, and it also gives the value 1 at an A_1 point (Remark that an A_1 point lies outside Σ.) Moreover $j(f) < \infty$. [Pe 1 5.5]

Theorem: [Pe 1] I 7.16
Let $(f,\Sigma,0)$ *be given as in* (1.1) *and assume* $(F,X,0)$ *is a deformation of* $(f,\Sigma,0)$ *over a regular analytic base* $(S,0)$.
Then there exist representatives $\mathcal{X}$ *of* $(X,0)$, $\mathcal{S}$ *of* $(S,0)$ *and an open neighbourhood* U *of* 0 *in* $\mathbb{C}$ *such that*

$$j(f) = \sum j(f_s, p)$$

for all $s \in \mathcal{S}$, *where the summation is taken over all* $p \in f_s^{-1}(U) \cap \mathcal{X}$.

Corollary: *If* $(F,X,0)$ *is a Morsification of* $(f,\Sigma,0)$ *then* $j(f) = \# A_1 + \# D_\infty$, *where* $\# A_1$ *and* $\# D_\infty$ *are as in* (1.3).

(1.5) For the number of D_∞ points Pellikaan uses the fact that $f \in I^2$ in the case that Σ is a complete intersection [Pe 1] I 1.9. Let $I = (g_1, \ldots, g_n)$ with $g_1, \ldots, g_n$ a regular sequence. Then we can write $f = \sum h_{ij} g_i g_j$, with $h_{ij} = h_{ji}$. Then Pellikaan proves:

Theorem: [Pe 1] I 7.18:

$$\# D_\infty = \dim_{\mathbb{C}}(\mathcal{O}/(I + \det(h_{ij})) =: h(f).$$

The purpose of this paper is to define a natural generalization of the number of D_∞ points, which we call the virtual number of D_∞ points, $VD_\infty(f)$, and to prove the corresponding "continuity" theorem:

Theorem: (see (2.5))
Consider the same situation as in theorem (1.4). *Then:*

$$VD_\infty(f) = \sum VD_\infty(f_s, p)$$

for all $s \in \mathcal{S}$, *where the summation is over all* $p \in \Sigma_s \cap \mathfrak{X}$.

This generalizes theorem (1.5) of Pellikaan, because $VD_\infty(D_\infty) = 1$ and $VD_\infty(A_\infty) = 0$, see (2.3). The definition of $VD_\infty(f)$ will be given with the help of vector fields which annihilate f. This reflects the fact that we consider non-isolated singularities. There are "non-trivial" vector fields which annihilate f, due to the the fact that the partial derivatives of f do not form a regular sequence, contrary to the isolated singularity case.
Surprisingly it turns out that the virtual number of D_∞ points can be negative (see (2.3), (2.7), (2.8)). For instance, if f = xyz, then $VD_\infty(f) = -2$. This shows in particular that f cannot have a Morsification.

(1.6) With the help of the invariants VD_∞, j(f) and $\mu(\Sigma)$, we can give a formula for the Euler characteristic of the Milnor fibre M of f, partially generalizing Siersma's result (1.3). If $b_i(M)$ denotes the i-th Betti number of the reduced homology of the Milnor fibre M we get the following (by now expected) theorem:

Theorem: (3.2)

$$b_n(M) - b_{n-1}(M) = j(f) + VD_\infty(f) + \mu(\Sigma) - 1$$

The equation should be understood that the left hand side, which is topological, can (at least in principle) computed by the right hand side, which is algebraic. The proof goes along the lines of Pellikaan [Pe 2], using a general result of Iomdin [Io]. Kato and Matsumoto proved that in our case M is (n-2) connected. For more information about non-isolated singularities we refer to Lê [Lê 1] and Randell [Ra].

(1.7) It should be remarked that the definition of the virtual number of D_∞ points is not very well suited for computations. For surfaces there will appear another formula in a forthcoming paper of Van Straten and the author [J-S]. In a subsequent paper A.J. de Jong and the author will show that the local invariant VD_∞ adds up to a gobal invariant of a divisor in a smooth manifold, where the divisor has a one dimensional singular locus and transverse A_1 singularities. This formula generalizes a classical formula about the number of ordinary singularities on surfaces in $\mathbb{P}^3$.

As we only work with germs throughout the paper, we will sometimes skip the notation of a germ.

(1.8) The author would like to thank F.-O Schreyer for many discussions and G.-M. Greuel for advices. Furthermore thanks to R. Pellikaan for comments.

§2 The virtual number of D_∞ points

(2.1) In this section, we want to introduce the virtual number of D_∞ points, and prove it's main property, the continuity under deformation. So let $(f,\Sigma,0)$ be given as in (1.1). Let $\mathcal{O}_\Sigma = \mathcal{O}/I$, where I is the ideal defining the reduced singular locus. Let Θ_f be the holomorphic vector fields along the fibres of f, see [Lo] 6.A. Then $\Theta_f \otimes \mathcal{O}_\Sigma \subseteq \Theta_Y$, where $Y = (f=0)$. Because $\Sigma \subseteq Y$ is the singular locus we have $\Theta_{Y|\Sigma} \subseteq \Theta_\Sigma$. To see this take an element $\xi \in \Theta_Y = \mathrm{Der}\,(\mathcal{O}_Y, \mathcal{O}_Y)$. We associate to it a vector field $\xi' \in \Theta_\Sigma = \mathrm{Der}(\mathcal{O}_\Sigma, \mathcal{O}_\Sigma)$ simply by defining $\xi'(\bar{f}) = \xi(f)$, where $f \in \mathcal{O}_Y$ maps to $\bar{f} \in \mathcal{O}_\Sigma$. This definition is independent of the lift f, if one shows that $\xi(I) \subseteq I$. But to show this, one has only to check it outside zero (after taking suitable representatives), i.e. for the A_∞ singularity, for which it is easily checked. We denote

$$\Theta(f) := \Theta_{f|\Sigma} \subseteq \Theta_\Sigma.$$

Moreover there is an inclusion $\Theta_\Sigma \subseteq \Theta_{\tilde{\Sigma}}$, where $\tilde{\Sigma}$ is the normalization of Σ, see [De]. We denote by $\delta(\Sigma)$ the δ-invariant of Σ, i.e. $\delta(\Sigma) = \dim\,(\mathcal{O}_{\tilde{\Sigma}}/\mathcal{O}_\Sigma)$.

Definition:

Let $(f,\Sigma,0)$ be as in (1.1).*Then*

$$VD_\infty(f) := \dim\,(\Theta_{\tilde{\Sigma}}/\Theta(f)) - 3\delta(\Sigma).$$

We call $VD_\infty(f)$ the virtual number of D_∞ points of f.

(2.2) We mention the following theorem, which gives us in principle a methopd to calculate $\Theta(f)$, and will be used in the proof of theorem (2.5).

Theorem: *Let* R *be a ring,* $J = (f_0,\dots,f_n)$ *be an ideal, with* $\mathrm{grade}(J) := \min\{i : \mathrm{Ext}^i_R(R/J,R) \neq 0\} = n$. *Suppose* $\mathrm{rad}(J) = I$, *with* $\mathrm{grade}(I/J) = n + 1$. *Consider the following diagram:*

$$\begin{tikzcd}
 & 0 \arrow[r] & G_n \arrow[r] & {} \arrow[r] & G_1 \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0 \\
0 \arrow[r] & \Lambda^{n+1}R^{n+1} \arrow[r] \arrow[u] & \Lambda^n R^{n+1} \arrow[r] \arrow[u, "\varphi_n"'] & \dots \arrow[r] & \Lambda^1 R^{n+1} \arrow[r] \arrow[u, "\varphi_1"'] & R \arrow[r] \arrow[u] & R/J \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where G. *is a projective resolution of* R/I *and the lower row is the Koszul comlex on* $\underline{f} = (f_0,\dots,f_n)$. *The maps* φ_i *extend the identity* $\mathrm{id}: R \longrightarrow R$. *Dualizing this diagram with respect to* R *and taking homology gives a map:* $\varphi_n^*: \mathrm{Ext}^n_R\,(R/I,R) \longrightarrow H_1(\underline{f},R)$, *where the last group denotes the first Koszul homology group. Then* φ_n^* *is an isomorphism.*

Proof: See [Pe 1] II 4.1 □

We apply this theorem to $(f,\Sigma,0)$ as in (1.1). We take $R = \mathcal{O}_{n+1}$, $I = \mathcal{I}(\Sigma)$, $f_i = \partial_i f$. Then we see that $H_1(\underline{f},R)$ is isomorphic to the dualizing sheaf ω_Σ of Σ. Moreover we can identify $H_1(\underline{f},R)$ with $\Theta(f)$ by sending a representative $(\alpha_0,\dots,\alpha_n)$ to the vector field $\sum \alpha_i\partial_i$. This map is well defined because if $(\alpha_0,\dots,\alpha_n) \in H_1(\underline{f},R)$ we have that $\sum \alpha_i f_i = \sum \alpha_i\partial_i f = 0$. Thus we get indeed a vector field which annihilates f. Moreover if $(\alpha_0,\dots,\alpha_n)$ represents the zero class in $H_1(\underline{f},R)$, then the vector field $\sum \alpha_i\partial_i$ is zero on Σ by the definition of the maps in the Koszul complex and the definition of the singular $\Sigma(f)$ of f.Finally, if the restriction of the vector field $\sum \alpha_i\partial_i$ to Σ is zero, then $(\alpha_0,\dots,\alpha_n)$ represents the zero class in $H_1(\underline{f},R)$. This is simply checked for the A_∞ singularity, and therefore (by taking suitable representatives again) it is zero outside zero. Because $H_1(\underline{f},R)$ is isomorphic to an Ext, it has no torsion. Therefore $(\alpha_0,\dots,\alpha_n)$ represents the zero class in $H_1(\underline{f},R)$.

(2.3) **Examples:**

1) A_∞: $f = \sum_{i=1}^n x_i^2$. In this case $\Theta_{\tilde{\Sigma}} = \Theta(f)$, $\delta(\Sigma) = 0$. Thus $VD_\infty(A_\infty) = 0$. This shows in particular that for all $(f,\Sigma,0)$ as in (1.1) that $VD_\infty(f) < \infty$.

2) D_∞: $f = x_0x_1^2 + \sum_{i=2}^n x_i^2$. Here $\Theta_{\tilde{\Sigma}} = \mathcal{O}_\Sigma\partial_0$, $\Theta(f) = \mathcal{O}_\Sigma x_0\partial_0$, $\delta(\Sigma) = 0$. Hence $VD_\infty(D_\infty) = 1$.

3) $T_{\infty,\infty,\infty}$: $f = xyz$, $I = (yz,xz,xy)$. Now $\Theta(f) = \mathcal{O}_\Sigma(x\partial_x - y\partial_y, z\partial_z - y\partial_y)$, $\Theta_{\tilde{\Sigma}} = \mathcal{O}_{\tilde{\Sigma}}(\partial_x, \partial_y, \partial_z)$, $\delta(\Sigma) = 2$. Hence $VD_\infty(T_{\infty,\infty,\infty}) = 4 - 6 = -2$.

(2.4) In case Σ is a complete intersection, it will follow from theorem (2.5) and Pellikaans theorem (1.5) that $h(f) = VD_\infty(f)$. (For the definition of $h(f)$ see (1.5).) We will give another argument. Let $I = (g_1,\dots,g_n)$, with $g_1,\dots,g_n$ a regular sequence. Let ϑ be the vector field dual to $dg_1\wedge\dots\wedge dg_n$, ie $\iota_\vartheta(dx_0\wedge\dots\wedge dx_n) = dg_1\wedge\dots\wedge dg_n$, where ι_ϑ denotes contraction with ϑ. By Van Straten's calculation [St] (3.4). $\Theta(f)$ is generated as $\mathcal{O}_\Sigma$- module by $h\Theta$, where $h = \det(h_{ij})$, see(1.5). Σ is a complete intersection, so the dualizing sheaf ω_Σ is $(dx_0\wedge\dots\wedge dx_n/dg_1\wedge\dots\wedge dg_n)\mathcal{O}_\Sigma$ (see [Se]). So we get $\Theta(f) = \mathrm{Hom}(h^{-1}\omega_\Sigma, \mathcal{O}_\Sigma)$.

Now let c_Σ: $\Omega^1_\Sigma \longrightarrow \omega_\Sigma$ be the canonical class morphism. By local duality we get: $\dim(\Theta_\Sigma/\Theta(f) = \dim(h^{-1}\omega_\Sigma/c_\Sigma\Omega^1_\Sigma)$. Consider the following inclusions:

$$c_\Sigma\Omega^1_\Sigma \subseteq \omega_\Sigma \subseteq h^{-1}\omega_\Sigma$$
$$\Theta(f) \subseteq \Theta_\Sigma \subseteq \Theta_{\tilde{\Sigma}}$$

By using these inclusions and the definition of the virtual number of D_∞ points we get $VD_\infty(f) = \dim(\Theta_{\tilde{\Sigma}}/\Theta(f)) - 3\delta(\Sigma) = \dim(\Theta_{\tilde{\Sigma}}/\Theta_\Sigma) +$ $+ \dim(h^{-1}\omega_\Sigma/\omega_\Sigma) + \dim(\omega_\Sigma/c_\Sigma\Omega^1_\Sigma) - 3\delta(\Sigma)$. But we have the following results about curves:

1) $\dim(\Theta_{\tilde{\Sigma}}/\Theta_\Sigma) = 3\delta(\Sigma) - \tau(\Sigma)$, ($\tau$ is the Tjurina number), because Σ is smoothable and non obstructed [Tj], and we have Deligne's formula for the dimension of a smoothing component [De].

2) $\dim(h^{-1}\omega_\Sigma/\omega_\Sigma) = \dim(\omega_\Sigma/h\omega_\Sigma) = \dim(\mathcal{O}_\Sigma/h\mathcal{O}_\Sigma) = h(f)$, because $\omega_\Sigma \cong \mathcal{O}_\Sigma$.

3) $\dim(\mathcal{O}_\Sigma/c_\Sigma\Omega^1_\Sigma) = \tau(\Sigma)$, see [B-G] (6.1.6) and [Gr 1].

Therefore by filling in we get that $VD_{\infty}(f) = h(f)$.

(2.5) We now come to the most important property of the virtual number of D_{∞} points.

Theorem: *Let* $(f,\Sigma,0)$ *be as in* (1.1) *and let* $(F,X,0)$ *be a deformation over a regular one dimensional base* $(S,0)$. *Then there exist representatives* $\mathcal{X}$ *of* $(X,0)$, $\mathcal{S}$ *of* $(S,0)$, *such that:*

$$VD_{\infty}(f) = \sum_{p\in\Sigma_s\cap\mathcal{X}} VD_{\infty}(f_s,p)$$

for all $s \in \mathcal{S}$, *i.e. the total virtual number of* D_{∞} *points is constant during a deformation.*

Proof: We apply theorem (2.2) to $\tilde{I} \subseteq \tilde{\mathcal{O}} := \mathcal{O}_{\mathbb{C}^{n+1}\times S}$ and the sequence $\underline{F} = (F_0,\dots,F_n) = (\partial_0F,\dots,\partial_nF)$. Then we get:

$$\omega_{X/S} = \mathrm{Ext}^n_{\tilde{\mathcal{O}}}(\tilde{\mathcal{O}}/\tilde{I},\tilde{\mathcal{O}}) \cong H_1(\underline{F},\tilde{\mathcal{O}})$$

where $\omega_{X/S}$ is the relative dualizing sheaf. One shows as in (2.2), that $H_1(\underline{F},\tilde{\mathcal{O}}) \cong \Theta(F)$, where $\Theta(F)$ are relative vector fields which annihilate F, restricted to $\mathcal{X} = V(\tilde{I})$. Because $\omega_{X/S}\otimes\mathcal{O}_{\Sigma} \cong \omega_{\Sigma}$ [B-G] (4.1.3) we get $\Theta(F)\otimes\mathcal{O}_{\Sigma} = \Theta(f)$. This shows that $\Theta(f) \subseteq \Theta_{X/S}\otimes\mathcal{O}_{\Sigma}$, the sheaf of vector fields on Σ which can be lifted to relative vector fields of $g\colon X \longrightarrow S$. We have (if s is a local parameter for $(\mathcal{S},0)$):

1) $\Theta_{X/S} \xrightarrow{\cdot s} \Theta_{X/S}$ is injective (by construction)

2) $\Theta(F) \longrightarrow \Theta_{X/S}$ is injective (trivial)

3) $\Theta(F)\otimes\mathcal{O}_{\Sigma} = \Theta(f) \longrightarrow \Theta_{X/S}\otimes\mathcal{O}_{\Sigma}$ is injective (see above).

It follows from the snake lemma that s is not a zero divisor of $\Theta_{X/S}/\Theta(F)$. Thus $g_*\Theta_{X/S}/g_*\Theta(F)$ is a flat $\mathcal{O}_S$ module, hence free, because S is regular. This gives us the statement:

$$\dim(\Theta_{X/S}\otimes\mathcal{O}_{\Sigma}/\Theta(f)) = \sum_{p\in\Sigma_s\cap\mathcal{X}} \dim(\Theta_{X/S}\otimes\mathcal{O}_{\Sigma_s,p}/\ \Theta(f_s,p))$$

The theorem follows by filling in the formula of the following lemma in the definition of the virtual number of D_{∞} points.

(2.6) **Lemma:**

let g: $(X,0) \longrightarrow (S,0)$ *be a deformation of* $(\Sigma,0)$ *over a one dimensional regular base* $(S,0)$. *Let* $\tilde{\Sigma}_s$ *be the normalization of* Σ_s.

Then: $\dim(\Theta_{\tilde{\Sigma}}/\Theta_{X/S}\otimes\mathcal{O}_{\Sigma}) = \sum_{p\in\Sigma_s\cap\mathcal{X}}\dim(\Theta_{\tilde{\Sigma}_s,p}/\Theta_{X/S}\otimes\mathcal{O}_{\Sigma_s,p}) +$
$$+3(\delta(\Sigma) - \sum_{p\in\Sigma_s\cap\mathcal{X}}\delta(\Sigma_s,p))$$
for all $s \in \mathcal{S}$, $\mathcal{X}$, $\mathcal{S}$ *sufficiently small.*

Proof: The case that g is a smoothing is treated in Greuel and Looijenga [G-L] p.273, who proved it in order to give a local proof of Deligne's formula about the dimension of a smoothing component of Σ [De]. The general case easily follows from this. One takes $n: \tilde{X} \longrightarrow X$ to be the normalization of X. One lifts the relative vector fields $\Theta_{X/S}$ to $\tilde{X}$, and get the inclusion:

$\Theta_{X/S} \longrightarrow \Theta_{\tilde{X}/S}$. By the same formal argument as in (2.5) one gets that $(gn)_*\Theta_{\tilde{X}/S}/(gn)_*\Theta_{X/S}$ is a free $\mathcal{O}_S$-module. But now one has that $gn: \tilde{X} \longrightarrow S$ is a smoothing of $\tilde{X}_0 := (gn)^{-1}(0)$. [B-G] (4.1.4). Therefore by using the result for a smoothing we get the formula:
$$\dim(\Theta_{\tilde{\Sigma}}/\Theta_{X/S}\otimes\mathcal{O}_{\Sigma}) = \sum_{p\in\Sigma_s\cap\mathcal{X}}\dim(\Theta_{\tilde{\Sigma}_s,p}/\Theta_{X/S}\otimes\mathcal{O}_{\Sigma_s,p} + 3\delta(\tilde{X}_0).$$

By Lejeune, Lê and Teissier [L-L-T] (see also [B-G] (4.1.4)) we have that $\delta(\tilde{X}_0) = \delta(\Sigma) - \sum\delta(\Sigma_s,p)$. □

(2.7 **Example:** (R.Pellikaan [Pe 1] I (7.7) (7.22).
Let $f: (\mathbb{C}^3,0) \longrightarrow (\mathbb{C},0)$, $f = (xy)^2 + (2xz)^2 + (yz)^2$. Then the singular locus consists of the three coordinate axes. Consider the following two deformations:

$F1 = (yz + sy - sz)^2 + (2xz + sx + - sz)^2 + (xy + sx - sy)^2$

$F2 = f + sxyz$

Then for generic values of s:

1) $f1_s$ has 4 D_∞ points and 6 A_1 points, Σ_s is smooth.
2) $f2_s$ has 6 D_∞ points, 4 A_1 points and one $T_{\infty,\infty,\infty}$ point.

It follows from theorem (2.5) that $VD_\infty(T_{\infty,\infty,\infty}) = -2$ (see the examples (2.3)). Remark that also the Jacobi number j(f) is constant under deformation.

(2.8) D. Mond [Mo] studied germs of mappings $\varphi: (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^3,0)$ and the deformation theory of this mapping. The image of φ is a hypersurface and under suitable finiteness assumptions it will have transverse A_1 singularities. Let the image of φ be given by the equation $f = 0$, and let Σ be the singular locus of f. Deforming the map φ gives a deformation of $(f,\Sigma,0)$ [J-S] (4.3). For a general deformation of φ one will have only C D_∞ points and T $T_{\infty,\infty,\infty}$ points in a general fibre (see [Mo]). Hence we get that $VD_\infty(f) = C - 2T$. We consider the following example:

$$\varphi(u,v) = (u,\ uv + v^{3k-1},\ v^3)$$

An equation for the image is:

$$f = -y^3 + x^3z + 3xyz^k + z^{3k-1} = 0$$

Here $C = 2$, and $T = k - 1$. Therefore $VD_\infty(f) = -2k + 4$. It can indeed be checked that: $\dim(\Theta_{\tilde{\Sigma}}/\Theta(f)) = 4k - 2$ and $\delta(\Sigma) = 2k - 2$.

§3 About the Milnor fibre

(3.1) Again let $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a germ of a holomorphic function with a one dimensional critical locus Σ and transverse type A_1. We choose ε and δ small, such that they satisfy the Milnor construction, i.e. $f^{-1}(t)$ is transversal to the sphere of radius ε in $\mathbb{C}^{n+1}$ for all t with $\|t\| < \delta$. Let B_ε be the closed ball of radius ε in $\mathbb{C}^{n+1}$. We will study the topology of the so called Milnor fibre $M := f^{-1}(t)\cap B_\varepsilon$. This definition is independent of the chosen t (see [Mi]). Let us recall from [Pe 3] that if Σ is smoothable and $f \in I^2$ ($\Sigma = V(I)$) then f always has a Morsification. This gives us a lot of functions to which the theorem of Siersma (1.3) can be applied. We can combine Siersma's result with the following result of Lê and Saito [L-S]:

Theorem:
Let $f\colon(\mathbb{C}^3,0) \longrightarrow (\mathbb{C},0)$, with a one dimensional singular locus and transverse type A_1. Then the fundamental group of the Milnor fibre of f is free abelian, with number of generators one less than the number of analytic components of $\{f = 0\}$ near the origin.

Corollary:
Let $(f,\Sigma,0)$ be as above. Then if

1) # { *irreducible components of* f=0} > 2 *or*
2) # { *irreducible components of* f=0} = 2 *and* $VD_\infty(f) \neq 0$,
then $(f,\Sigma,0)$ *does not have a Morsification.*

Remark: In 2) the condition $VD_\infty(f) \neq 0$ is necessary. In fact using theorem (1.5) we can if Σ is a complete intersection, systematically give examples of $(f,\Sigma,0)$ with $VD_\infty(f) = 0$, $\{f=0\}$ having two irreducible components and $(f,\Sigma,0)$ having a Morsification.

(3.2) Although for n = 2 there is a nice statement about the first Betti number of the Milnor fibre, for general n there seems not much to be known about the (n-1)st Betti number, although Van Straten [St] gives a method which will probably always work. A general result about the Euler characteristic of the Milnor fibre is much easier to establish.

Theorem:
Let $f:(\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ *be a germ of a holomorphic function with a one dimensional singular locus* Σ *and transverse type* A_1. *Let* M *be the Milnor fibre of* f. *Then:*

$$b_n(M) - b_{n-1}(M) = j(f) + VD_\infty(f) + \mu(\Sigma) - 1$$

(We take reduced homology with coefficients in $\mathbb{Z}$.*)*

In case Σ is a complete intersection Pellikaan [Pe 2] proved this theorem by using the formula of Iomdin [Io], which compares the Euler characteristic of the Milnor fibre M of f with the Euler characteristic of the Milnor fibre of 'nearby' isolated singularities. We follow Pellikaan's proof.

(3.3) We first note the following special case of Iomdin's theorem [Io].

Theorem: *Let* $f:\ (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ *be holomorphic with a one dimensional critical locus* Σ *and transverse type* A_1. *Choose the coordinates of* $\mathbb{C}^{n+1}$ *in such a way, that* $x := x_0$ *is a generic hyperplane. Then* $f_k := f + x^{k+1}/k+1$ *has an isolated singularity. Let*

M_k *be the Milnor fibre of* f_k. *Then:*

$$b_n(M) - b_{n-1}(M) = b_n(M_k) - (k+1)\, m(\Sigma)$$

if $k>>0$ *and* $m(\Sigma) = \dim(\mathcal{O}_\Sigma/(x))$.

(3.4) We denote by I the ideal of Σ, J the Jacobi ideal of f, and J_k the Jacobi ideal of f_k. It is very well-known that (see [Mi]):

$$b_n(M_k) = \dim(\mathcal{O}/J_k)$$

This is what we need to calculate in order to apply Iomdin's theorem.

Lemma:

$b_n(M_k) = \dim(\mathcal{O}/J_k) + km(\Sigma) + \dim(J/I\cap J_k) + \dim(I/J)$ *if* $k>>0$.

Proof: Consider the following two exact sequences:

$$0 \longrightarrow I/I\cap J_k \longrightarrow \mathcal{O}/J_k \longrightarrow \mathcal{O}/I+J_k \longrightarrow 0$$

$$0 \longrightarrow J/I\cap J_k \longrightarrow I/I\cap J_k \longrightarrow I/J \longrightarrow 0$$

and observe that $I + J_k = I + (x^k)$ if $k>>0$.□

(3.5) Because $\dim(I/J) < \infty$, we have that $\dim(J/I\cap J_k)$ does not depend on k if k>>0. In fact one has:

Lemma: $J/I\cap J_k \cong \mathcal{O}/(I + (f_1,\dots,f_n):f_0)$ *where* $f_i = \partial_i f$, $i=0,\dots,n$.

Proof: See [Pe 2] proof of (3.5). □

(3.6) Pellikaan proceeds by explicitely calculating $\dim(\mathcal{O}/(I + (f_1,\dots,f_n):f_0)$ in case Σ is a complete intersection. Because this is of course impossible in the general case, we proceed in a different way. Consider the map:

$$h\colon \Theta_\Sigma \longrightarrow \mathcal{O}_\Sigma$$
$$\xi \longmapsto \iota_\xi(dx)$$

where ι_ξ denotes contraction with ξ. In coordinates: if $\xi = \sum \alpha_i\partial_i$, then $h(\xi) = \alpha_0$. Because x is generic, we have that h is injective, since dx is nonzero on every branch of Σ (outside zero and taking suitable representatives), and there are no torsion vector fields. Furthermore, by definition, h maps $\Theta(f)$ to $(f_1,\dots,f_n):f_0 + I$. Hence we get:

$$\dim(\Theta_\Sigma/\Theta(f)) = \dim(\mathcal{O}/(I + (f_1,\dots,f_n):f_0)) - \dim(\mathcal{O}_\Sigma/h(\Theta_\Sigma)),$$

By definition of the virtual number of D_∞ points we have moreover:

$\dim(\Theta_\Sigma/\Theta(f)) = 3\delta(\Sigma) + VD_\infty(f) - m_1(\Sigma)$, where $m_1(\Sigma) := \dim(\Theta_{\tilde\Sigma}/\Theta_\Sigma)$.

(3.7) **Lemma:**

$$\dim(\mathcal{O}_\Sigma/h(\Theta_\Sigma)) = m_1(\Sigma) - \delta(\Sigma) - r(\Sigma) + m(\Sigma),$$

where $r(\Sigma)$ *denotes the number of branches of* Σ.

Proof: (cf. [Gr 2]). Choosing local uniforming parameters for Σ, say $t_1,\dots,t_r$ we get the $\mathcal{O}_{\tilde\Sigma}$ isomorphism:

$$\phi\colon \Omega_{\tilde\Sigma}\otimes K \cong K$$

$$(dt_1,\dots,dt_r) \longmapsto (t_1,\dots,t_r)$$

where K is the total quotient ring of $\mathcal{O}_\Sigma$. We define $N := \phi\circ j(\Omega_\Sigma)$, $j\colon \Omega_\Sigma \longrightarrow \Omega_{\tilde\Sigma}$ being the canonical morphism. Because $\Theta_\Sigma = \mathrm{Hom}(\Omega_\Sigma, \mathcal{O}_\Sigma)$ and $\Theta_{\tilde\Sigma} = \mathrm{Hom}(\Omega_{\tilde\Sigma}, \mathcal{O}_{\tilde\Sigma})$ we have the identifications:

$$\Theta_\Sigma = \mathcal{O}_\Sigma : N,\ \Theta_{\tilde\Sigma} = \mathcal{O}_{\tilde\Sigma} : \tilde{m},$$

$\tilde{m}$ denoting the Jacobson radical of $\mathcal{O}_{\tilde\Sigma}$. It is easily checked that under these identifications, the map h of (3.6) becomes multiplication with the element $h := \sum_{i=1}^{r(\Sigma)}(\partial x/\partial t_i)t_i$. Now consider the inclusions:

$$h(\mathcal{O}_\Sigma : N) \subseteq h(\mathcal{O}_{\tilde\Sigma} : \tilde{m}) \subseteq \mathcal{O}_{\tilde\Sigma} : \tilde{m}$$

$$h(\mathcal{O}_\Sigma : N) \subseteq \mathcal{O}_\Sigma \subseteq \mathcal{O}_{\tilde\Sigma} \subseteq \mathcal{O}_{\tilde\Sigma} : \tilde{m}$$

By the first inclusion we get: $\dim(\mathcal{O}_{\tilde\Sigma} : \tilde{m}/h(\mathcal{O}_\Sigma : N)) =$

$$= \dim(h(\mathcal{O}_{\tilde\Sigma} : \tilde{m})/h(\mathcal{O}_\Sigma : N)) + \dim(\mathcal{O}_{\tilde\Sigma} : \tilde{m}) =$$

$$= \dim(\mathcal{O}_{\tilde\Sigma} : \tilde{m}/\mathcal{O}_\Sigma : N) + m(\Sigma) = m_1(\Sigma) + m(\Sigma).$$

By this result and the second inclusion:

$$\dim(\mathcal{O}_\Sigma/h(\Theta_\Sigma) = \dim(\mathcal{O}_\Sigma/h(\mathcal{O}_\Sigma : N)) = \dim(\mathcal{O}_{\tilde\Sigma} : \tilde{m}/h(\mathcal{O}_\Sigma : N)) -$$

$$- \dim(\mathcal{O}_{\tilde\Sigma}/\mathcal{O}_\Sigma) - \dim(\mathcal{O}_{\tilde\Sigma} : \tilde{m}/\mathcal{O}_{\tilde\Sigma}) = m_1(\Sigma) + m(\Sigma) - \delta(\Sigma) - r(\Sigma). \ \square$$

(3.8) Now we combine Iomdin's formula (3.3) with our calculations. The result is:

$$b_n(M) - b_{n-1}(M) = j(f) + VD_\infty(f) + 2\delta(\Sigma) - r(\Sigma).$$

But because $\mu(\Sigma) = 2\delta(\Sigma) - r(\Sigma) + 1$ [B-G], we can rewrite this to:

$$b_n(M) - b_{n-1}(M) = j(f) + VD_\infty(f) + \mu(\Sigma) - 1,$$

which gives theorem (3.2).

(3.9) **Example:** $f = (x^2y^2 + y^3 + z^2)x$. Then $\mu(\Sigma) = 3$, $j(f) = 4$ and $VD_\infty(f) = 0$. By Lê and Saito [L-S] $\pi_1(M) \cong \mathbb{Z}$, because $f = 0$ has two irreducible components. It follows by theorem (4.3) of [J-S] that $(f,\Sigma,0)$ does not have a Morsification. We get however:

$b_2(M) - 1 = 4 + 0 + 3 - 1 = 6$. It even follows from a theorem of Siersma [Si 2] (6.1) that M is homotopy equivalent to a wedge of 7 two-spheres wedged with a circle.

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THE VIRTUAL NUMBER OF D_∞ POINTS II

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1. In [Jo], the second author defined an invariant, the virtual number of D_∞ points, for special types of non-isolated hypersurface singularities. Now, let X be a compact variety with a one dimensional singular locus Σ, such that X has generic A_∞ along Σ. In this paper we prove that if X is a divisor in a compact complex manifold Y then the total virtual number of D_∞ points of X can be computed by global invariants of X and Y.

2. We recall from [Jo] the definition of the virtual number of D_∞ points. Let f: $(\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a germ of a holomorphic function with a one dimensional singular locus $(\Sigma,0)$ and generic A_∞ singularities along Σ. Throughout the paper we give Σ its reduced structure. Let $\Theta_f = \{\vartheta \in \Theta_{\mathbb{C}^{n+1},0}: \vartheta(f) = 0$, and $\Theta(f) = \Theta_{f|\Sigma}$. Let n: $(\tilde{\Sigma},0) \longrightarrow (\Sigma,0)$ be the normalization map and $\delta(\Sigma,0)$ be the δ- invariant of $(\Sigma,0)$. Since any derivation of $\mathcal{O}_\Sigma$ lifts uniquely to a derivation of $\mathcal{O}_{\tilde{\Sigma}}$ (see [De]) we get inclusions:

$$\Theta(f) \subseteq \Theta_{\Sigma,0} \subseteq \Theta_{\tilde{\Sigma},0}.$$

Definition: *The virtual number of* D_∞ *points of* f *at* 0 *is:*

$$VD_\infty(f,0) = \dim(\Theta_{\tilde{\Sigma}}/\Theta(f)) - 3\delta(\Sigma,0).$$

3. Let Y be a compact complex manifold of dimension n+1 and X $\subset$ Y a divisor with singular locus Σ and as above: $\dim(\Sigma) = 1$ and generic A_∞ singularities along Σ. For all $p \in \Sigma$ let (X,p) be the germ of the non-isolated singularity of X at p. Let K_Y be the

canonical divisor on Y. The purpose of this paper is to prove the following:

Theorem:

$$\sum_{p\in\Sigma} VD_\infty(X,p) = \langle 2K_Y + n\,X,\ \Sigma\rangle + 4\,\chi(\Sigma,\mathcal{O}_\Sigma)$$

where $\langle\ ,\ \rangle$ denotes the intersection product on Y.

4. As an example we take $Y = \mathbb{P}^3$ and X a surface with only ordinary singularities, say C D_∞ points (local equation $xy^2 + z^2 = 0$) and T triple points (local equation $xyz = 0$). It is easily calculated (cf. [Jo] (2.3)) that the virtual number of D_∞ points of a D_∞ point is one and of a triple point -2. This shows that the theorem specializes to:

$$C - 2T = (2\deg(X) - 8)\deg(\Sigma) + 4\chi(\Sigma,\mathcal{O}_\Sigma).$$

This formula is a rewritten version of a well-known classical formula, cf. [G-H] p. 628. Our proof, however, is completely different.

5. Consider the same situation as in 2. Let $K.(f)$ be the Koszul complex on the partial derivatives of f, i.e.

$$0 \longrightarrow \Lambda^{n+1}\mathcal{T}_{\mathbb{C}^{n+1},0} \longrightarrow \ldots \longrightarrow \mathcal{T}_{\mathbb{C}^{n+1},0} \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1},0} \longrightarrow 0$$

where $\mathcal{T}_{\mathbb{C}^{n+1},0}$ is the tangentsheaf of $(\mathbb{C}^{n+1},0)$. It is remarked in [Jo] (2.2) that the obvious map:

$$H_1(K.(f)) \longrightarrow \Theta_\Sigma \qquad (1)$$

gives an isomorphism of $H_1(K.(f))$ with $\Theta(f)$. Moreover $H_1(K.(f))$ is (non canonically) isomorphic to $\omega_{\Sigma,0}$ (see [Pe] II 4.1).

Let $E.(f))$ be the cokernel of multiplication of f on $K.(f)$:

$$0 \longrightarrow K.(f) \xrightarrow{\cdot f} K.(f) \longrightarrow E.(f) \longrightarrow 0 \qquad (2)$$

We have $H_i(K.(f)) = 0$ for $i > 1$ [Pe] so the long exact cohomology sequence of (2) gives $H_i(E.(f)) = 0$ for $i > 2$ and an exact sequence:

$$0 \longrightarrow H_2(E.(f)) \longrightarrow H_1(K.(f)) \xrightarrow{\cdot f} H_1(K.(f)) \longrightarrow \ldots$$

But as $H_1(K.(f))$ is annihilated by f (because $H_1(K.(f)) \cong \omega_{\Sigma,0}$) we get an isomorphism:

$$H_2(E.(f)) \xrightarrow{\sim} H_1(K.(f)) \qquad (3)$$

The crucial point of the proof of the theorem is that although we do not know how to globalize $K.(f)$ we do know how to globalize

$E.(f)$.

6. Let the notation be as in 3. Let $\mathcal{T} = \mathcal{T}_{Y|X}$ be the tangentsheaf of Y restricted to X and N be the normal sheaf of X in Y. Then we have the canonical map: $\mathcal{T} \longrightarrow N$, which gives us a section $d_X \in \Gamma(X, \Omega^1_{Y|X}\otimes N)$. We form the Koszul complex associated to this section:

$$\mathcal{E}.: \quad 0 \longrightarrow \Lambda^{n+1}\mathcal{T}\otimes N^{-n-1} \longrightarrow \dots \longrightarrow \mathcal{T}\otimes N^{-1} \longrightarrow 0$$

For any points $p \in X$, choose a small open ball $U \subset Y$ around p, and a local equation f for $X\cap U$. This determines a trivialization of N on U and so we get an isomorphism:

$$\mathcal{E}.\otimes N^2{}_{|U} \cong E.(f) \tag{4}$$

By composing this map on homology with the maps (3) and (1) we get a map:

$$\Psi_{f,U}: \ H_2(\mathcal{E}.\otimes N^2{}_{|U}) \longrightarrow \Theta_{\Sigma|U}$$

which gives an isomorphism of $H_2(\mathcal{E}.\otimes N^2{}_{|U})$ with $\Theta(f)$. We claim that $\Psi_{f,U}$ does not depend on choises made and so we get a global map:

$$\Psi: \ H_2(\mathcal{E}.\otimes N^2) \longrightarrow \Theta_\Sigma$$

Proof of the claim:

We have the complex $\mathcal{E}.\otimes N^2$:

$$\dots \longrightarrow \Lambda^3\mathcal{T}\otimes N \longrightarrow \Lambda^2\mathcal{T} \longrightarrow \mathcal{T}\otimes N \longrightarrow \dots$$

Given a section $\vartheta_1\wedge\vartheta_2 \in \Gamma(U, \Lambda^2(\mathcal{T}_Y))$ one has:

$$\Psi_{f,U}(\vartheta_1\wedge\vartheta_2{}_{|X}) = f^{-1}(\iota_{\vartheta_1\wedge\vartheta_2}(\omega_f))_{|\Sigma}$$

where ι denotes contraction and we have written:

$$d_X = \omega_f\otimes f^{-1}{}_{|U\cap X} \in \Omega^1_Y\otimes\mathcal{O}_Y(X)_{|U\cap X}$$

But if we now have another local equation g of $U\cap X$ then:

$$\omega_g = \omega_f g f^{-1}$$

so it follows that

$$\Psi_{f,U}(\vartheta_1\wedge\vartheta_2{}_{|X}) = \Psi_{g,U}(\vartheta_1\wedge\vartheta_2{}_{|X}). \ \square$$

Definition: $\Theta(X) := \mathrm{Im}(\Psi)$.

Remark that on any U as above we have $\Theta(X)_{|U} = \Theta(f)$.

7. **Proposition:** There exists a canonical isomorphism:

$$\varphi\colon H^{n-1}(\mathcal{H}om(\mathcal{E}.,\omega_X)) \cong \omega_X$$

Proof: Let $\mathcal{F}.$ (with $\mathcal{F}_0 = \mathcal{O}_X$) be a locally free resolution of $\mathcal{O}_\Sigma$ such that there exists a diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varphi_2} & & \uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ \cdots & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \end{array}$$

where the φ_i extend the identity on $\mathcal{O}_X$. (This we can do if for example X is projective. If it is not possible the we just do it on open sets U where it is possible and then glue the local maps φ_U together to get the global map φ.) By dualizing with respect to ω_X and taking cohomology we get a map:

$$\varphi^{-1} := \varphi^*_{n-1}\colon \mathcal{E}xt^{n-1}(\mathcal{O}_\Sigma,\omega_X) \longrightarrow H^{n-1}(\mathcal{H}om(\mathcal{E}.,\omega_X)).$$

By [Pe] II 4.1 this is an isomorphism. Morover:

$$\mathcal{E}xt^{n-1}(\mathcal{O}_\Sigma,\omega_X) = \omega_\Sigma,$$

so taking the inverse of φ^{-1} gives the map φ. □

8. **Lemma:** $\Theta(X) \cong \omega_\Sigma \otimes \omega_Y \otimes \mathcal{N}^{-n}$.

Proof: As $\mathcal{E}.$ is a Koszul complex, $\mathcal{H}om(\mathcal{E}.,\omega_X)$ can be rewritten as:

$$\mathcal{E}.\otimes(\Lambda^{n+1}\mathcal{I})^{-1}\otimes\mathcal{N}^{n+1}\otimes\omega_X$$

Furthermore $(\Lambda^{n+1}\mathcal{I})^{-1} = \omega_{Y|X}$ and by the adjunction formula $\omega_Y\otimes \mathcal{N} = \omega_X$. Hence by 6. and 7. we have:

$$\omega_\Sigma \cong H^{n-1}(\mathcal{H}om(\mathcal{E}.,\omega_X)) = H_2(\mathcal{E}.\otimes\mathcal{N}^2)\otimes\omega_Y\otimes\mathcal{N}^n \cong \Theta(X)\otimes\omega_Y\otimes\mathcal{N}^n. \square$$

9. **Proof of the theorem:**

The proof of the theorem is now simply calculating Euler characteristics. Define Q by the following exact sequence:

$$0 \longrightarrow \Theta(X) \longrightarrow n_*\Theta_{\tilde{\Sigma}} \longrightarrow Q \longrightarrow 0$$

Q is a skyscraper sheaf [Jo] (2.3), so

$$H^0(\Sigma,Q) = \chi(\Theta_{\tilde{\Sigma}}) - \chi(\Theta(X))$$

By remark 6. and the definition of the virtual number of D_∞ points:

$$\textstyle\sum_{p\in\Sigma} VD_\infty(X,p) = H^0(\Sigma,Q) - 3\sum_{p\in\Sigma}\delta(\Sigma,p).$$

By Riemann-Roch:

$$\chi(\Theta_{\tilde{\Sigma}}) = 3\chi(\mathcal{O}_{\tilde{\Sigma}})$$

and because $\chi(\mathcal{O}_\Sigma) = \chi(\mathcal{O}_{\tilde{\Sigma}}) - \sum_{p\in\Sigma}\delta(\Sigma,p)$ we get:

$$\sum_{p\in\Sigma} VD_\infty(X,p) = 3\chi(\mathcal{O}_\Sigma) - \chi(\Theta(X)).$$

By lemma 8:

$$\sum_{p\in\Sigma} VD_\infty(X,p) = 3\chi(\mathcal{O}_\Sigma) - \chi(\omega_\Sigma \otimes \omega_Y^{-2} \otimes \mathcal{N}^{-n})$$

and thus:

$$\sum_{p\in\Sigma} VD_\infty(X,p) = \langle 2K_Y + nX,\ \Sigma\rangle + \chi(\Sigma,\mathcal{O}_\Sigma) + 3\chi(\Sigma,\mathcal{O}_\Sigma).$$

This concludes the proof of the theorem. □

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CHAPTER THREE

Deformations of Non – Isolated Singularities.

by

Theo de Jong and Duco van Straten.

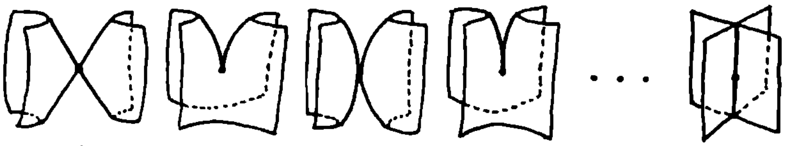
Contents :

Introduction.

Given a germ $X = (X,p)$ of an analytic space with an isolated singular point p, one has a *semi-universal deformation* $\mathfrak{X} \longrightarrow B$. It has the property that all flat families over a space Z with X as special fibre are induced by a map $Z \longrightarrow B$, which is unique on the level of tangent spaces. The space B and the deformation $\mathfrak{X} \longrightarrow B$ are unique, but only up to non-canonical isomorphism (see [Gra], [Pa], [Schl1], [Bi]). The space B is called the *base space (of the semi-universal deformation) of X.* If X is a hypersurface, or more generally a complete intersection, then B is smooth (see [Tj1]). If X is Cohen-Macaulay of codimension 2 (i.e. embdim(X) - dim(X) = 2), then B is also smooth (see [Scha]). In general however B may be singular and even have components of different dimensions. The simplest example is the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$, due to Pinkham (see [Pi]). In this case B consists of two smooth components, one of dimension three and one of dimension one, intersecting each other transversally.

In general it is very hard to compute the base space B for a given singularity X. Only recently the base spaces of all cyclic quotient singularities were determined by Arndt (see [Arn]). Usually the first step in the construction of B consists of finding T^1_X , the set of first order deformations of X, which can be naturally identified with the Zariski tangent space of B. This space T^1_X has received much attention, in particular in the case that X is a normal surface singularity. Using a resolution of X one can try to compute T^1_X in terms of of resolution data (see [La], [Wa1]). This has been rather succesful for rational singularities (see [Ri], [B-K]).

For a hypersurface singularity T^1_X can be identified with $\mathcal{O}_X/J(f)$, where J(f) is the the ideal generated by the partial derivatives of f, and f = 0 is a defining equation of X. One easily sees that T^1_X is finite dimensional if and only if X has an isolated singular point. When X does not have an isolated singular point, it is natural to look for a *special class* of deformations, namely the class of deformations for which the singular locus of X is deformed flatly (and stays inside the singular locus of the deformed X!). Under appropriate circumstances one can hope for a finite dimensional base space, because the infinite dimensionality of T^1_X is caused by the 'opening up' of the singularities transverse to the singular locus. A good example to keep in mind is the A_k-series of deformations of the A_∞-singularity:



$$A_1 \longleftarrow A_2 \longleftarrow A_3 \longleftarrow A_4 \longleftarrow \cdots A_\infty$$

In his thesis, Pellikaan [Pe1] (see also [Pe2]) started with a theory along these lines, extending the case that the singular locus is smooth and one dimensional, which was considered by Siersma (see [Si]). Pellikaan's main results however concern the case that the singular locus Σ is a complete intersection or the case that Σ is not deformed at all. If Σ is a complete intersection, then the base space of the functor considered by Pellikaan is smooth (see also §3.C). That this is not always the case, can be seen by the following beautiful example of Pellikaan (see [Pe 2], ex.2.4). This is the example of $f = (yz)^2 + (zx)^2 + (xy)^2$. Here the (reduced) singular locus Σ is described by the ideal $I = (\Delta_1, \Delta_2, \Delta_3) = (yz, zx, xy)$, so Σ consists of the coordinate axes in $\mathbb{C}^3$. He gives two types of deformations. First of all, one can deform the curve Σ, giving a deformed ideal $\tilde{I} = (\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3)$. Then obviously $F = (\tilde{\Delta}_1)^2 + (\tilde{\Delta}_2)^2 + (\tilde{\Delta}_3)^2$ gives a deformation in the above sense. Because one knows that $\dim T^1_\Sigma = 3$, and in this case Σ is Cohen-Macaulay of codimension 2, one gets a deformation over a smooth 3 dimensional space. Another deformation, over a smooth curve with parameter s, is given by $F = f + s.xyz$. Here Σ is not deformed at all, and the deformation is also admissible in the above sense, because $xyz = 0$ has also the coordinate axes as singular locus. These two types of deformations are 'essentially' all admissible deformations of f. So we get a family over a space B which is the same as the base space in Pinkham's example we mentioned above. This is not a coincidence. Because our space X, defined by $f = 0$, has singularities in codimension one, it is not normal. Now the normalization $\tilde{X}$ of X is precisely the cone over the rational normal curve in $\mathbb{P}^4$. Moreover, the total space of the deformation over the one dimensional component of $\tilde{X}$ can be identified with the cone over the Veronese surface in $\mathbb{P}^5$ (see [Pi]). It is known that a 'generic' projection in $\mathbb{P}^3$ of the Veronese surface is the Steiner Roman surface, described in homogeneous coordinates by the equation $(yz)^2 + (zx)^2 + (xy)^2 + sxyz = 0$ (see [S-R], pp.128-135). This indeed corresponds exactly to the second type of deformation of X described above.

In general, we consider a normal surface singularity $\tilde{X}$, embedded in some high dimensional space. When we now project $\tilde{X}$ down to $\mathbb{C}^3$, we get a hypersurface X as image. This hypersurface X will in general have a curve Σ as singular locus. For a 'generic' projection X will be *weakly normal* or what is the same, X will have *transverse A_1 - singularities* meaning that in a general point $q \in \Sigma$ one has $(X,q) \approx A_\infty$ (i.e. Σ will be an ordinary double curve). Conversely, given a weakly normal surface $X \subset \mathbb{C}^3$, one can take the normalization to get an $\tilde{X}$. Now the statement is that the functor of admissible deformations of X is equivalent to the deformation functor of the diagram $\tilde{X} \longrightarrow X$. (see §4.) As the deformation theory of $\tilde{X}$ is not 'essentially' different from the deformation theory of the diagram $\tilde{X} \longrightarrow X$, this implies that all pathologies occurring in the deformation theory of normal surfaces are reflected in the deformation theory of non-isolated hypersurface singularities in $\mathbb{C}^3$.

The purpose of this paper is to develop the (formal) theory of *admissible deformations* of non-isolated singularities, as intended above. We give a short overview of what to expect. In § 0. we treat some algebraic results. This paragraph should be used as a reference, and can therefore be skipped on first reading. In § 1. we introduce the functor of admissible deformations $Def(\Sigma,X)$ of a singularity X with a subspace Σ of the singular locus of X as a sub-functor of the deformations of the diagram $\Sigma \hookrightarrow X$, and investigate the Schlessinger conditions. In § 2. we consider the problem whether the natural forgetful transformation $Def(\Sigma,X) \longrightarrow Def(X)$ is injective. In § 3. we develop the infinitesimal deformation theory for non-isolated hypersurfaces. We determine the tangent space $T^1(\Sigma,X)$ of $Def(\Sigma,X)$ and identify the obstruction space $T^2(\Sigma,X)$. In § 4. we prove the above mentioned equivalence between $Def(\Sigma,X)$ and $Def(\tilde{X} \longrightarrow X)$. In § 5. finally we give examples and appplications. For a weakly normal surface X in $\mathbb{C}^3$ with normalization $\tilde{X}$ we give a formula for $T^1_{\tilde{X}}$ in terms of X only. Furthermore we prove a theorem about the dimension of the smoothing components of a normal surface singularity $\tilde{X}$ in terms of the number of triple points $(xyz = 0)$ occurring in the deformation of X. Finally, in § 6. we determine, up to a smooth factor, the base spaces of the semi-universal deformation of all rational quadruple points. This proof reflects our experience that to understand the deformation theory of normal surface singularities it is essential to study the double locus Σ of a projection in $\mathbb{C}^3$ (c.f. (3.28)).

It should be stressed that although we try to formulate our results as general as possible, the case that interests us most and which we always *have in mind* is the case where X is an an analytic germ of a weakly normal surface in $\mathbb{C}^3$ and Σ is the singular locus of X, with its *reduced* structure. So along our way we are always willing to make any assumption on X and Σ as long as it applies to this case. For some results simpler arguments can be given in the special case we have in mind, which we usually for reasons of organization and clearity have avoided.

Conventions.

By a *space* we always mean an analytic space germ or the spectrum of a local ring. Typical names for spaces are X, Y, T, Σ, etc, for rings R, P, S, etc. When we say that 'X_S is a space over S' we mean that X_S is a space with a map to Spec(S) or to S, depending on whether S is a *ring* (this is usually the case) or a *space* . In such a relative situation we do simply write X_S/S in cases where one usually should write $X_S/\mathrm{Spec}(S)$. Although we are not completely systematic in this respect, we do not expect any confusion to arise.

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In this preliminary paragraph we formulate some results from local commutative algebra which will be of use in § 2 and § 4 of this paper. Lacking a comprehensive reference and for convenience of the reader we include proofs. The results are centered around the interaction between the notion of flatness, base change properties of Ext, Cohen - Macaulay properties and duality. This paragraph may be skipped on a first reading.

In the sequel we adopt the following conventions:

* S is a noetherian local ring with maximal ideal m_S and residue field $k = S/m_S$.
* R is a noetherian local S - algebra
* We also consider R - modules M, N, . . . , and we always assume them to be finitely generated.
* When we put a bar over a module M we always mean $\overline{M} = \overline{R} \otimes_R M = k \otimes_S M$, where $\overline{R} = k \otimes_S R$.
* For some results we assume that R is a quotient of flat S - algebra P such that $\overline{P}$ is a regular local ring.

Proposition (0.1) :

Let M and N be two S-flat R - modules. Consider the natural mappings

$$\varphi_i \ : \ \mathrm{Ext}^i_R(M,N)\otimes k \longrightarrow \mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{N})$$

Then: i) If φ_i is surjective, then it is an isomorphism.

ii) If φ_i and φ_{i-1} are surjective, then $\mathrm{Ext}^i_R(M,N)$ is S-flat.

proof: This is a slight variation on the 'Cohomology and base change theorem' (see [Ha], pp 282-290). Let $F_{\cdot} \longrightarrow M$ be an R - free resolution of M. Then the complex $N^{\cdot} := \mathrm{Hom}_R(F_{\cdot}, N)$ consists of finitely generated R - modules which are S - flat and has $\mathrm{Ext}^i_R(M,N)$ as cohomology groups. We have $\bar{N}^{\cdot} = \mathrm{Hom}_R(F_{\cdot}, \bar{N}) = \mathrm{Hom}_{\bar{R}}(\bar{F}_{\cdot}, \bar{N})$ and as M is S - flat, $\bar{M}$ is resolved by the complex $\bar{F}_{\cdot}$. Hence, the cohomology of the complex $\bar{N}^{\cdot}$ computes $\mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N})$. Consider now the functor T^i on S - modules:

$$T^i : A \longmapsto T^i(A) := H^i(N^{\cdot} \otimes_S A).$$

By the usual arguments one has:

* T^i left exact $\Leftrightarrow W^i := \mathrm{Coker}(N^{i-1} \longrightarrow N^i)$ is S - flat.
* T^i right exact $\Leftrightarrow \varphi_i : T^i(S) \otimes_S A \twoheadrightarrow T^i(A)$, for all A

  $\Leftrightarrow \varphi_i$ is an isomorphism for all A.

But also: T^i right exact $\Leftrightarrow T^{i+1}$ left exact $\Leftrightarrow W^{i+1}$ S - flat $\Leftrightarrow$ (local criterion for flatness, [Ma], pp. 145-149) $W^{i+1} \otimes m_S \hookrightarrow W^{i+1}$ $\Leftrightarrow T^{i+1}(m_S) \hookrightarrow T^{i+1}(S) \Leftrightarrow T^i(S) \otimes k \twoheadrightarrow T^i(k)$. From this the proposition follows. ⊠

Corollary (0.2) :

Under the same assumptions as in proposition (0.1) one has :

i) If $\mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N}) = 0$ then $\mathrm{Ext}^i_R(M,N) = 0$.

ii) If $\mathrm{Ext}^k_{\bar{R}}(\bar{M},\bar{N}) = 0$ for $k = i-1$ and $k = i+1$, then

$\mathrm{Ext}^i_R(M,N)$ is S-flat and $\mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N}) = \mathrm{Ext}^i_R(M,N) \otimes_S k$.

proof: Statement i) follows easily from (0.1) together with the lemma of Nakayama, as we know that the modules $\mathrm{Ext}^i_R(M,N)$ are finitely generated modules over R. For statement ii) note as φ_{i+1} is surjective, the functor T^{i+1} is right exact and as $T^{i+1}(S) = 0$ by i), we find that $T^{i+1}(m_S) = 0$ and hence φ_i is surjective, so by (0.1) ii) we are done. ⊠

Lemma (0.3) :

Let M be any finitely generated R - module and N be an S - flat R - module.

Then : $\mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N}) = 0$ for $i = 0, 1, \ldots, p$ implies

$\mathrm{Ext}^i_R(M,N) = 0$ for $i = 0, 1, \ldots, p$.

proof: By [Ma], thm 28, p. 100 we have that $\mathrm{Ext}^i_{\bar{R}}(\bar{M},\bar{N}) = 0$ for $i = 0, 1, \ldots, p$ is equivalent to the existence of elements $\bar{x}_i \in \bar{R}$, $i = 0, 1, \ldots, p$ such that

i) $\bar{x}_i \in \mathrm{Ann}_{\bar{R}}(\bar{M})$

ii) the $\bar{x}_i$ form a regular $\bar{N}$ - sequence.

Now let $m_1, m_2, \ldots, m_t$ be R - generators for M and $x \in R$ any lift of one of the $\bar{x}_i$. Then $x.M \subset \mathfrak{m}_S .M$, so $\det(x.I - B).M = 0$, (where B is any matrix of x. with respect to the generators m_i of M) by Cramers rule. As the entries of the matrix B are in the maximal ideal, we see that the elements $y_i := \det(x_i .I - B) \in \mathrm{Ann}_R(M)$ project to $\bar{x}_i^{\,t}$. As these form a regular $\bar{N}$ - sequence and N is S- flat, we have that the y_i form a regular N - sequence (see [Ma], pp. 150-151). Hence the lemma follows by application of [Ma], thm. 28 again. ⊠

Definition (0.4) :

Let R and S as above and let M be an R - module. We say that:

* M is *Cohen - Macaulay over S* (CM over S) if and only if
  i) $\bar{M}$ is a Cohen - Macaulay $\bar{R}$ - module (i.e. $\dim_{\bar{R}}(\bar{M}) = \mathrm{depth}_{\bar{R}}(\bar{M})$).
  ii) M is S - flat.

  We call $d := \dim_{\bar{R}}(\bar{M})$ the *dimension* and $c := \dim(\bar{R}) - d$ the *codimension* of M over S. If $c = 0$ we say that M is maximal Cohen-Macaulay over S (M is MCM over S).

* R is *regular over S* if and only if

  i) $\bar{R}$ is a regular local ring.

  ii) R is S - flat.

  We call $N := \dim(\bar{R})$ the *relative dimension* of R over S.

For a local ring that is regular over S we will use the symbol P.
We call $\omega_{P/S} := P$ the dualizing module of P over S.

Proposition (0.5) :

Let P be regular over S of relative dimension N. For an S - flat P - module M the following conditions are equivalent:

i) M is CM over S of codimension c.

ii) $\mathrm{Ext}^i_{\bar{P}}(\bar{M}, \omega_{\bar{P}}) = 0$ for $i \neq c$.

proof : First assume i). The relation between depth and local cohomology (see [Gro], cor. 3.10, p.47) tells us that $H^i_m(\bar{M})=0$ for $i < N - c$. Then the local duality theorem for the regular local ring $\bar{P}$ (see [Gro], thm 6.3, p. 85) states that $H^i_m(\bar{M})$ is (Matlis-)dual to $\mathrm{Ext}^{N-i}_{\bar{P}}(\bar{M}, \omega_{\bar{P}})$. Hence we have $\mathrm{Ext}^k_{\bar{P}}(\bar{M}, \omega_{\bar{P}}) = 0$ for $k > c$. The vanishing of the lower Ext's follows by Ischebek's lemma ([Ma], (15.E), p.104), because the dimension of $\bar{M}$ is $N-c$ and the depth of $\omega_{\bar{P}}$ is N. Hence we get ii). To get i) from ii) one just reverses the above steps. ⊠

Definition (0.6) :

Let P be regular over S and let M be a P -module which is CM over S of codimension c. The *dual module* of M is defined to be

$$M^{\vee} := \mathrm{Ext}^c_P(M, \omega_{P/S}) .$$

An S - algebra R is called *embeddable* if R is the quotient of a ring P which is regular over S. If R is Cohen - Macaulay over S of codimension c considered as a P - module, we define the *dualizing module* to be $\omega_{R/S} := R^{\vee} = \mathrm{Ext}^c_P(R, \omega_{P/S})$.

Proposition (0.7) :

Let P be regular over S and let M be CM over S of codimension c. Then one has:

i) $\mathrm{Ext}_P^k(M,\omega_{P/S}) = 0$ for $k \neq c$.

ii) the dual module $M^\vee$ is S - flat .

iii) $\overline{(M^\vee)} = (\bar{M})^\vee$.

proof : Combine (0.5) with (0.2). In fact, for an S - flat module M, the Cohen - Macaulay property is equivalent to the above three properties. ⊠

Remark (0.8) :

By the change-of-rings spectral sequence (see [C-E], p. 349)

$$E_2^{pq} = \mathrm{Ext}_R^p(M,\mathrm{Ext}_P^q(R,N)) \Rightarrow \mathrm{Ext}_P^{p+q}(M,N)$$

one can relate Ext's over different rings. If R is embeddable and CM over S of codimension c as a P - module then one has an isomorphism

$$\mathrm{Ext}_R^p(M,\omega_{R/S}) = \mathrm{Ext}_P^{p+c}(M,\omega_{P/S})$$

for any R - module M. This also shows that $\omega_{R/S}$ is essentially independent of the choice of P in (0.6).

Corollary (0.9) :

Let R be embeddable and CM over S of codimension c.
Then one has:

i) Propositions (0.5) and (0.7) hold for P replaced by R.

ii) If M is CM over S of codimension e considered as an R - module, then M is CM over S of codimension e+c considered as a P - module.

Proposition (0.10) :

Let R be embeddable and CM over S and let M be an R - module which is CM over S of codimension c. Then one has:

i) $M^{\vee} = \mathrm{Ext}^{c}_{R}(M,\omega_{R/S})$ is also CM over S of codimension c.

ii) There is a natural isomorphism $M \longrightarrow (M^{\vee})^{\vee}$.

proof : It is not hard to see that by using (0.9) one can reduce to the case that R = P , P regular over S. Consider a minimal free resolution $F_{.} \longrightarrow M$ of M over P. Because M is S - flat, the complex $\overline{F}_{.}$ is a minimal free resolution for $\overline{M}$. Because $\overline{M}$ is Cohen - Macaulay of codimension c over the regular local ring $\overline{P}$, we conclude by the Auslander -Buchsbaum formula (see [A-B], thm 2.3, p.397) that the length of the complex $F_{.}$ is exactly c, i.e. the resolution looks like

$$0 \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

When we apply $\mathrm{Hom}_P(-,P)$ we get the complex

$$0 \longrightarrow F_0^{\vee} \longrightarrow F_1^{\vee} \longrightarrow \ldots \longrightarrow F_{c-1}^{\vee} \longrightarrow F_c^{\vee} \longrightarrow M^{\vee} \longrightarrow 0$$

where $F^{\vee}_{.} = \mathrm{Hom}_P(F_{.},P)$. By (0.7) this last complex is exact, hence we get a free resolution of $M^{\vee}$. As we already know that $M^{\vee}$ is S - flat by (0.7) one can reverse the steps and the result follows. ⊠

Consider a (germ of a) space X and a subspace Σ contained in the singular locus of X. In this paragraph we define a certain subfunctor of the functor of deformations of the inclusion map of Σ in X, which consists of deformations for which the deformed Σ stays inside the singular locus of the deformed X. First we have to define an appropriate structure on the critical locus of a map (and hence on the singular locus of a space). Let $X \longrightarrow S$ be a flat mapping of relative dimension n.

Definition (1.1) :

The *critical locus* $\mathcal{C} := \mathcal{C}_{X/S}$ is the locus defined by $F_n(\Omega^1_{X/S})$, the n-th Fitting ideal of the sheaf of relative Kähler one-forms. The *critical space* is $\mathcal{C}$ together with $\mathcal{O}_{\mathcal{C}} := \mathcal{O}_X/F_n(\Omega^1_{X/S})$ as structure sheaf.

This definition can be found in [Te], def.2.5, p.587. It is natural to consider the critical space again as a space over S. One of the reasons to define the critical space in this way is because of the following

Property (1.2) :

The formation of the critical space commutes with base-change. This comes down to a simple property of Fitting ideals (see [Te], p.570)

Definition (1.3) :

• A *diagram over S* is a triple (Σ_S , X_S , i), where Σ_S and X_S are spaces over S and $i : \Sigma_S \longrightarrow X_S$ is a map over S. Usually we will be sloppy and say that $\Sigma_S \longrightarrow X_S$ is a diagram over S, without even mentioning the map.

• A *morphism of diagrams* is defined in the obvious way.

* A diagram $\Sigma_S \longrightarrow X_S$ over S is said to be *admissible*, if the map $i: \Sigma_S \longrightarrow X_S$ factorizes over the inclusion map $\mathcal{C}_{X_S/S} \hookrightarrow X_S$.

* A morphism between admissible diagrams over S is just a morphism of the underlying diagrams over S.

* Let $\Sigma \longrightarrow X$ be a diagram over Spec(k), k a field. Let S be the spectrum of a local ring with residue field k. A diagram $\Sigma_S \longrightarrow X_S$ over S is said to be a *deformation* of the diagram $\Sigma \longrightarrow X$ iff :

i) Σ_S and X_S are flat over S.

ii) $\left(\Sigma \longrightarrow X\right) \approx \left(\Sigma_S \longrightarrow X_S\right) \times_S \mathrm{Spec}(k)$

* A deformation $\Sigma_S \longrightarrow X_S$ of $\Sigma \longrightarrow X$ is called *admissible* or is said to be an *admissible deformation* if the diagram $\Sigma_S \longrightarrow X_S$ is admissible.

Let $\mathbf{C}$ denote the category of local noetherian k - algebras with residue field k. It has a full subcategory $\mathbf{C}_a$ consisting of Artinian algebras. Let **Set** denote the category of sets.

Definition (1.4) :

Let $\Sigma \longrightarrow X$ be an admissible diagram over Spec(k).

$$\text{The functor} \quad \mathbf{C} \longrightarrow \mathbf{Set}$$
$$S \longmapsto \left\{\text{isomorphism classes of deformations of } \Sigma \longrightarrow X \text{ over } S\right\}$$

is called the *functor of deformations of the diagram* $\Sigma \longrightarrow X$ and is denoted by $\mathrm{Def}(\Sigma \longrightarrow X)$.

$$\text{The functor} \quad \mathbf{C} \longrightarrow \mathbf{Set}$$
$$S \longmapsto \left\{\text{isomorphism classes of admissible deformations of } \Sigma \longrightarrow X \text{ over } S\right\}$$

is called the *functor of admissible deformations* and is denoted by $\mathrm{Def}(\Sigma, X)$.

We remark that the base-change property (1.2) is needed to make Def(Σ,X) into a functor. Remark further that Def(Σ, X) is a subfunctor of Def($\Sigma \longrightarrow$ X). We do not make a notational distinction between these functors and their restriction to the subcategory $\mathbf{C_a}$.

We recall that if T' and T ϵ Ob($\mathbf{C_a}$) then α: T' $\longrightarrow$ T is called a *simple surjection* if α is a surjection and Ker(α) is a principal ideal in T' with $\mathrm{Ker}(\alpha).m_{T'} = 0$, where $m_{T'}$ is the maximal ideal of T'. (see [Schl 1], 1.2).

Lemma (1.5):

The functor F := Def($\Sigma \longrightarrow$X) is semi-homogeneous, i.e. three of the four *Schlessinger conditions* are satisfied:

i) $F(k) = \{pt\}$

ii) $F(T'' \times_T T') \longrightarrow F(T'') \times_{F(T)} F(T')$ is surjective for every simple surjection $T' \longrightarrow T$ and every morphism $T'' \longrightarrow T$.

iii) $F(T' \times_k k[\varepsilon]/\varepsilon^2) \longrightarrow F(T') \times F(k[\varepsilon]/\varepsilon^2)$ is an isomorphism for all T'.

The proof is similar to [Schl1], 3.7 . In fact, for ii), if we are given deformations $\Sigma_S \longrightarrow X_S$, $\Sigma_{S'} \longrightarrow X_{S'}$ and $\Sigma_{S''} \longrightarrow X_{S''}$ (S = Spec(T) etc.) with

$$\left(\Sigma_{S'} \longrightarrow X_{S'}\right) \times_{S'} S \approx \left(\Sigma_{S''} \longrightarrow X_{S''}\right) \times_{S''} S \approx \left(\Sigma_S \longrightarrow X_S\right)$$

then the natural map

$$\left(\left(\Sigma_{S''} \amalg_{\Sigma_S} \Sigma_{S'}\right) \longrightarrow \left(X_{S''} \amalg_{X_S} X_{S'}\right)\right) \qquad (*)$$

gives a deformation of the diagram over $S' \times_S S''$ which restricts to the given deformations over S' and S".

Proposition (1.6):

Def(Σ, X) is a semi-homogeneous subfunctor of Def($\Sigma \longrightarrow$ X).

proof: Our definitions are casted in such a way that the proof just becomes a repetition of the proof that the functor of ***deformations with a singular section*** has an analogous property. That case corresponds to $\Sigma = \{pt\}$ and has been treated by Buchweitz (see [Bu], p.79). We keep the notation as above, but now we are given $\Sigma_S \longrightarrow X_S$, etc. , which are admissible. We have to show that the diagram (*) under (1.5) in fact is admissible. It is clear that the map (*) factorizes over

$$\mathcal{C}_{X_{S''}/S''} \amalg_{\mathcal{C}_{X_S/S}} \mathcal{C}_{X_{S'}/S'}$$

But by the base change property of the critical locus (1.2) there is a natural morphism

$$\mathcal{C}_{X_{S''}/S''} \amalg_{\mathcal{C}_{X_S/S}} \mathcal{C}_{X_{S'}/S'} \longrightarrow \mathcal{C}_{X_{S''} \amalg_{X_S} X_{S'} / S'' \times_S S'}$$

which gives us the factorization which shows the admissibility of (*). It is now clear that the result follows because $\mathrm{Def}(\Sigma \longrightarrow X)$ itself is a semi-homogeneous functor. ⊠

Corollary (1.7) :

If $T^1(\Sigma,X) := \mathrm{Def}(\Sigma, X)(k[\varepsilon]/(\varepsilon^2))$ is a finite dimensional vector space, then $\mathrm{Def}(\Sigma, X)$ satisfies the Schlessinger conditions; i.e. $\mathrm{Def}(\Sigma, X)$ has a ***hull*** (i.e. a there is a 'formal' semi-universal deformation).

proof : See [Schl 1], 2.11 . ⊠

Suppose that we have an admissible diagram $\Sigma \hookrightarrow X$ and an embedding of X in some smooth ambient space Y. Analogous to the functor $\mathrm{Def}(\Sigma,X)$ of admissible deformations one can define a functor $\mathrm{Embdef}(\Sigma,X)$ of admissible deformations which can be realized inside Y. It is of some importance to describe the relation between $\mathrm{Def}(\Sigma,X)$ and $\mathrm{Embdef}(\Sigma,X)$, because in practice one always describes X and Σ by *equations*, so an embedding is always implicit. We now shall make this relation more precice.

Definition (1.8) :

Let Y be a space smooth over k. An *embedded admissible diagram* (over Spec(k)) is a diagram $\Sigma \hookrightarrow X \hookrightarrow Y$ (over Spec(k)) such that $\Sigma \hookrightarrow X$ is admissible. An *embedded admissible deformation* over S is a diagram $\Sigma_S \hookrightarrow X_S \hookrightarrow Y_S \approx Y \times S$ over S such that $\Sigma_S \hookrightarrow X_S$ is an admissible deformation of $\Sigma \hookrightarrow X$. Morphisms between such objects are defined in the obvious way. The functor

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{Set} \\ S &\longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of embedded admissible} \\ \text{deformations of } \Sigma \hookrightarrow X \hookrightarrow Y \text{ over } S \end{array} \right\} \end{aligned}$$

is called the *functor of embedded admissible deformations* and is denoted by Embdef(Σ, X), the space Y being understood.

Lemma (1.9) :

The natural forgetful transformation

$$\mathrm{Embdef}(\Sigma,X) \longrightarrow \mathrm{Def}(\Sigma,X)$$

is smooth.

This statement is completely analogous to the corresponding statement about ordinary deformations. We omit the proof and refer to [Ar1] for further details.

In § 1. we introduced the functor $\mathrm{Def}(\Sigma, X)$ of admissible deformations, consisting of deformations of X together with a subspace Σ of the critical space $\mathfrak{C}$ of X. There is a natural forgetful transformation

$$\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X) .$$

In this section we formulate some conditions under which this is an injection, i.e. conditions under which $\mathrm{Def}(\Sigma, X)(S) \hookrightarrow \mathrm{Def}(X)(S)$ is injective for all S in the category $\mathbf{C}$. Intuitively, this should be the case when Σ can not 'move' inside $\mathfrak{C}$. One expects this to be the case when Σ and $\mathfrak{C}$ are equal at the generic points. The problem is to find a good functorial way to reconstruct Σ from X alone.. The conditions we find are probably unnecessarily strong, but they suffice for the applications we have in mind. We use some generalities from local algebra which can be found in § 0.

Lemma (2.1):

Let R and S be rings as in § 0. Consider an exact sequence of R - modules:

$$0 \longrightarrow N \longrightarrow A \longrightarrow M \longrightarrow 0$$

Assume that:

i) $\bar{M}$ is Cohen-Macaulay of codimension c.

ii) M is S - flat.

iii) $\mathrm{Ext}^i_{\bar{R}}(\bar{N}, \omega_{\bar{R}}) = 0$ for $i = 0, 1, \ldots, c$.

Then $M \approx (A^{\vee})^{\vee} := \mathrm{Ext}^c_R(\mathrm{Ext}^c_R(A, \omega_{R/S}), \omega_{R/S})$.

proof : By (0.4) M is CM over S of codimension c. By (0.3) we have that $\mathrm{Ext}^i_R(N, \omega_{R/S}) = 0$ for $i = 0, 1, \ldots, c$. When we take $\mathrm{Hom}_R(-, \omega_{R/S})$ of the above exact sequence we get $M^\vee \approx A^\vee$. Hence the lemma follows from (0.10). ⊠

The above lemma expresses the fact that if the 'difference' $\bar{N}$ between $\bar{A}$ and $\bar{M}$ is 'small', then a possible flat deformation of $\bar{M}$ to M is completely determined by A, even if A is not flat. For this to be true, one of course needs some purity of M, like the CM - assumption. We use this fact to prove the following theorem:

Proposition (2.2) :

Let $\Sigma \hookrightarrow X$ an admissible diagram over Spec(k) and let I be the ideal of Σ in $\mathcal{O}_X$. Assume that:

i) X is Cohen-Macaulay of dimension n.

ii) Σ is Cohen-Macaulay of codimension c in X.

iii) $\mathcal{E}xt^i_X(I/F_n(\Omega^1_X), \omega_X) = 0$ for $i = 0, 1, \ldots, c$.

Then the natural forgetful transformation

$$\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$$

is injective.

proof : Let $\Sigma_S \hookrightarrow X_S$ be an admissible deformation of $\Sigma \hookrightarrow X$ over S. Because $\Sigma_S \hookrightarrow \mathcal{C}_S$, where $\mathcal{C}_S$ is the critical space of X_S over S, we get an exact sequence of $\mathcal{O}_{X_S}$ - modules:

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{O}_{\mathcal{C}_S} \longrightarrow \mathcal{O}_{\Sigma_S} \longrightarrow 0$$

where $\mathcal{N} = I_S/F_n(\Omega^1_{X_S/S})$ (I_S the ideal of Σ_S in $\mathcal{O}_{X_S}$).

Our assumptions are of such a nature that we can apply lemma (2.1) to get $\mathcal{O}_{\Sigma_S} \approx (\mathcal{O}_{\mathcal{C}_S}{}^{\vee})^{\vee}$. Hence the arrow $\mathcal{O}_{X_S} \longrightarrow \mathcal{O}_{\Sigma_S}$ is naturally identified with the composition $\mathcal{O}_{X_S} \longrightarrow \mathcal{O}_{\mathcal{C}_S} \longrightarrow (\mathcal{O}_{\mathcal{C}_S}{}^{\vee})^{\vee}$. As the critical space $\mathcal{C}_S$ is determined in a canonical way by $X_S \longrightarrow S$, we see that we can reconstruct $\Sigma_S \hookrightarrow X_S$ from the map $X_S \longrightarrow S$ alone; i.e. $\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$ is injective. ⊠

Corollary (2.3) :

Let $\Sigma \hookrightarrow X$ be an admissible deformation over Spec(k). Assume that Σ and X are Cohen-Macaulay. If $\dim\big(\mathrm{Supp}(I/F_n(\Omega^1_X))\big) < \dim(\Sigma)$, then the transformation $\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$ is injective.

proof : This follows immediately from (2.2), because by local duality iii) is equivalent to $H^i_{\{0\}}(I/F_n(\Omega^1_X)) = 0$ for $i \geq \dim(\Sigma)$. As local cohomology of a module vanishes above the dimension of its support, we get the result. ⊠

In the case that X is a (germ of a) reduced hypersurface singularity, given by an equation of the form $f=0$, $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$, then

$$I/F_n(\Omega^1_X) = I/(f, J(f))$$

where $J(f) := (\partial_0 f, \partial_1 f, \ldots, \partial_n f)$ is the Jacobian ideal, generated by the partial derivatives $\partial_i f = \partial f/\partial x_i$. A further specialisation of (2.3) is the following.

Corollary (2.4) :

Let X be a hypersurface germ defined by $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ and let Σ be defined by an ideal $I \supset (f, J(f))$. Assume that:

i) Σ is Cohen-Macaulay of dimension ≥ 1

ii) $\dim_{\mathbb{C}}(I/(f, J(f)) < \infty$.

Then the forgetful transformation $\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$ is injective.

Remark (2.5) :

When the ideal I is *reduced* and the conditions of (2.4) apply, we say that X has (generically) transverse A_1 - singularities. In the above context of hypersurface singularities, Pellikaan [Pe] studied modules of the form $I/J(f)$. His results imply the following:

*Theorem (*Pellikaan, [Pe 4], thm. (3.3), (3.4), (3.5)*)*
Let $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ define a germ of a mapping $f: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$. Let Σ be defined by an ideal $I \supset J(f)$ such that $\dim_{\mathbb{C}}(I/J(f)) < \infty$. Assume that one of the following conditions hold:

i) Σ is Cohen-Macaulay and $\dim(\Sigma) = 1$.

ii) Σ is a complete intersection.

iii) Σ is Cohen-Macaulay of codimension 2.

Then for an admissable deformation of the mapping f and Σ (defined analoguous to (1.3)) $I/J(f)$ is *flat.*

Note however that the module $I/(f, J(f))$ can *not* expected to behave in a flat way. (c.f. μ and τ for an isolated hypersurface singularity.)

We conclude this section with an example that will also play a role in § 4. and which shows that $\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$ is not always injective.

Example (2.6) :

Let X be defined by $f = x^3 + y^2 \in k[x,y]$ ($\mathrm{char}(k) \neq 2,3$), so $J_f = (x^2, y)$. Let Σ be the subspace of the critical locus defined by $I = (x,y)$. Consider the trivial deformation of X over $k[\varepsilon]/(\varepsilon^2)$, defined by the same function f, but now considered in $k[\varepsilon, x, y]/(\varepsilon^2)$. Let $I_1 = (x,y) \subset k[\varepsilon,x,y]/(\varepsilon^2)$ and $I_2 = (x+\varepsilon, y) \subset k[\varepsilon,x,y]/(\varepsilon^2)$. Because $x^2 = (x+\varepsilon).(x-\varepsilon)$ we see that both I_1 and I_2 correspond to admissible deformations of the pair Σ, X. One can check that these elements are different in $\mathrm{Def}(\Sigma, X)(k[\varepsilon]/(\varepsilon^2))$, but map to the trivial deformation of X in $\mathrm{Def}(X)(k[\varepsilon]/(\varepsilon^2))$. Hence $\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$ is not injective in this example.

This paragraph is devoted to the study of the functor $\mathrm{Def}(\Sigma,X)$ in the case that X is a germ of a hypersurface singularity. This study is divided into three parts. In part A we study the vector space $T^1(\Sigma,X) := \mathrm{Def}(\Sigma,X)(k[\varepsilon]/(\varepsilon^2))$ of first order admissible deformations. Part B is devoted to the obstruction theory, i.e. conditions for extending a given deformation over a somewhat bigger base space. Under appropriate circumstances the obstruction space we get is a quotient of (the torsion subsheaf of) Ω^1_Σ, the Kähler differentials on Σ. We also prove a theorem that states roughly that the base space of the semi universal deformation space of $\mathrm{Def}(\Sigma,X)$ depends *mainly* on Σ. In part C we prove that if Σ is not obstructed, then the obstruction space is in fact an in general much smaller subspace of the quotient of $\mathrm{Tors}(\Omega^1_\Sigma)$ we got in B. A crucial rôle is played by the so called Hessian form. For computational purposes this does not seem to be to important, but for theoretical purposes it probably is. Further we give some formulas relating this Hessian to other invariants.

Notations and Conventions (3.1):

Throughout this paragraph X will denote a germ of a hypersurface in $(\mathbb{C}^{n+1},0)$, defined by an equation $f = 0$, $f \in \mathcal{O} := \mathcal{O}_{\mathbb{C}^{n+1},0} = \mathbb{C}\{x_0, x_1, \ldots, x_n\}$. In fact, as all our arguments will be algebraic in nature, we might as well replace $\mathcal{O}$ by any local ring which is smooth over a field k.

Σ will be a subspace of X, defined by an ideal $I \subset \mathcal{O}$.

Furthermore, we put

$$\textstyle\int I := \left\{ g \in \mathcal{O} \,\middle|\, (g, \partial_i g) \subset I \right\} \tag{1}$$

where $\partial_i g := \partial g / \partial x_i$ is the partial derivative of g with respect to x_i. $\int I$ is an ideal and is called the *primitive ideal* of I (see [Pe 3], def. 1.1) This notion of primitive ideal leads to a convenient formulation of the condition that Σ is contained in the critical locus of X. Clearly:

$$\Sigma \subset \mathcal{C}_X \quad \Leftrightarrow \quad f \in \int I$$

An alternative way to define $\int I$ is by the exact sequence

$$0 \longrightarrow I/\int I \xrightarrow{d} \Omega^1 \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0 \tag{2}$$

where $\Omega^1 := \Omega^1_{\mathbb{C}^{n+1},0}$

We choose generators Δ_i, $i = 1, 2, \ldots, m$ for the ideal I:

$$I = \left(\Delta_1, \Delta_2, \ldots, \Delta_m \right) \tag{3}$$

Because $f \in I$ ($\Sigma \subset X$) we can write:

$$f = \sum_{i=1}^{m} \alpha_i \cdot \Delta_i \tag{4}$$

Because $f \in \int I$, there are elements $\omega_i \in \Omega^1$, $i = 1, 2, \ldots, m$ such that

$$df = \sum_{i=1}^{m} \omega_i \cdot \Delta_i \tag{5}$$

In order to keep the notation as simple as possible we suppress all indices, i.e. we simply write $I = (\Delta)$, $f = \alpha.\Delta$ and $df = \omega.\Delta$ instead of (3), (4) and (5). We will extend this 'summation convention' without any further comment to new situations. To make this more precise, we choose a presentation of $\mathcal{O}_\Sigma$ as an $\mathcal{O}$ - module

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0 \tag{6}$$

where $\mathcal{F} := \mathcal{O}^m$ and $\mathcal{F} \longrightarrow \mathcal{O}$ is the map $e_i \longmapsto \Delta_i$. $\mathcal{R}$ is the module of *relations* between the generators Δ_i, i.e. $r \in \mathcal{R} \Leftrightarrow r.\Delta = 0$. Then α can be considered as an element of $\mathcal{F}$ and is determined by f modulo $\mathcal{R}$ and ω can be considered as an element of $\mathcal{F} \otimes \Omega^1$ and is determined modulo $\mathcal{R} \otimes \Omega^1$.

So we have: $f \in \int I \Leftrightarrow \exists\, \alpha, \omega \mid f = \alpha.\Delta$ and $df = \omega.\Delta$. We will use frequently the following equivalent form of this statement.

Lemma (3.2): $f \in \int I \Leftrightarrow \exists\, \alpha, \Gamma \mid f = \alpha.\Delta$ and $\alpha.d\Delta + \Gamma.\Delta = 0$.

proof: From $f = \alpha.\Delta$ we get $df = d\alpha.\Delta + \alpha.d\Delta$. As $df = \omega.\Delta$ we get $0 = \alpha.d\Delta + \Gamma.\Delta$, where $\Gamma = d\alpha - \omega$. ⊠

$\mathcal{A}$. First Order Deformations and $T^1(\Sigma, X)$.

Let Embdef(Σ, X) be the functor of admissible embedded deformations of Σ in X. There are two obvious forgetful transformations

$$P_X : \text{Embdef}(\Sigma, X) \longrightarrow \text{Embdef}(X)$$
$$P_\Sigma : \text{Embdef}(\Sigma, X) \longrightarrow \text{Embdef}(\Sigma)$$

We put
$$\mathcal{A} := \text{Embdef}(\Sigma, X)(S)$$
$$N_X := \text{Embdef}(X)(S)$$
$$N_\Sigma := \text{Embdef}(\Sigma)(S)$$

where $S = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Via the natural mappings $P_X : \mathcal{A} \longrightarrow N_X$ and $P_\Sigma : \mathcal{A} \longrightarrow N_\Sigma$ we can consider $\mathcal{A}$ as a subset of $N_\Sigma \times N_X$. We call $\mathcal{A}$ the set of *admissible pairs*. The set $P_\Sigma(\mathcal{A})$ we call the set of admissible normal vectors to Σ and $P_X(\mathcal{A})$ the set of admissible functions. All these sets have a natural $\mathbb{C}$-vector space structure.

Let us first briefly recall the description of the vector spaces $N_\Sigma := \text{Embdef}(\Sigma)(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ and $T^1_\Sigma := \text{Def}(\Sigma)(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ of first order (embedded) deformations (see also [Ar 1]). Let $\Sigma \subset \mathbb{C}^{n+1}$ be defined by an ideal $I = (\Delta) \subset \mathcal{O}$ and consider a flat deformation Σ_ε of Σ over Spec($\mathbb{C}[\varepsilon]/(\varepsilon^2)$). Then Σ_ε is defined by an ideal $I_\varepsilon = (\Delta_\varepsilon) = (\Delta + \varepsilon.\, n) \subset \mathcal{O}[\varepsilon]/(\varepsilon^2)$. Now flatness in terms of relations means that for all relations $r \in \mathcal{R}$ for (Δ) there is a relation $r_\varepsilon = r + \varepsilon.s$ for (Δ_ε), i.e. $r_\varepsilon \cdot \Delta_\varepsilon = 0$. Expanding the product and using $\varepsilon^2 = 0$ we see that an embedded first order deformation is given by an n such that for all $r \in \mathcal{R}$ one has $r.\, n + s.\Delta = 0$ for some s. So n can be considered as a homomorphism $I \longrightarrow \mathcal{O}_\Sigma$; $\Delta_i \longmapsto n_i$. (In the sequel we usually will not make a distinction between a normal vector n and its set of components (n_i)). From this it follows that

$$N_\Sigma = \text{Hom}(I, \mathcal{O}_\Sigma) = \text{Hom}_\Sigma(I/I^2, \mathcal{O}_\Sigma) \tag{7}$$

The $\mathcal{O}_\Sigma$ - module on the right hand side is called the *normal sheaf* (of Σ in the ambient space), being the dual of the conormal sheaf I/I^2.

The space of first order deformations is obtained from the space of first order embedded deformations by dividing out by the infinitesimal automorphisms. These are generated by the vector fields Θ on the ambient space: a $\vartheta \in \Theta$ gives rise to the homomorphism $\left(\Delta_i \longmapsto \vartheta(\Delta_i)\right) \in N_\Sigma$. Thus one sees that T^1_Σ sits in an exact sequence

$$0 \longrightarrow \Theta_\Sigma \longrightarrow \Theta \otimes \mathcal{O}_\Sigma \longrightarrow N_\Sigma \longrightarrow T^1_\Sigma \longrightarrow 0 \qquad (8)$$

which starts with the dual of the sequence (2).
The description of N_X and T^1_X for the hypersurface X is of course very easy: $N_X \approx \mathcal{O}_X = \mathcal{O}/(f)$ and $T^1_X \approx \mathcal{O}/(f, \partial_i f) = \mathcal{O}_{C_X}$.

Now we can describe the set $\mathcal{A}$ of admissible pairs as follows:

$$\mathcal{A} = \Big\{ (n, g) \in N_\Sigma \times N_X \,\Big|\, \exists\, \alpha_1 \in \mathcal{F},\ \omega_1 \in \mathcal{F} \otimes \Omega^1 \text{ such that}$$
$$1)\quad (f + \varepsilon.g) = (\alpha + \varepsilon.\alpha_1).(\Delta + \varepsilon.n)$$
$$2)\ d(f + \varepsilon.g) = (\omega + \varepsilon.\omega_1).(\Delta + \varepsilon.n) \Big\}$$

These conditions can be rewritten as

$$\exists\, \alpha_1 \in \mathcal{F},\ \omega_1 \in \mathcal{F} \otimes \Omega^1 \;\Big|\; \begin{array}{l} 1)\quad g = \alpha.n + \alpha_1.\Delta \\ 2)\ dg = \omega.n + \omega_1.\Delta \end{array} \qquad (9)$$

These expressions motivate the following definitions.

Definition (3.3) :

The α and the ω - map are given by:

$$\alpha_f : N_\Sigma \longrightarrow \mathcal{O}_\Sigma \quad ; \quad n \longmapsto \alpha.n$$
$$\omega_f : N_\Sigma \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma \; ; \quad n \longmapsto \omega.n$$

Furthermore we define the w - map by

$$w_f : N_\Sigma \longrightarrow \Omega^1_\Sigma \quad ; \quad n \longmapsto d(\alpha.n) - \omega.n = \alpha.dn + \Gamma.n$$

We usually omit the index f if no confusion is likely.

Lemma (3.4) :

The α, ω and w - map only depend on $f \in \int I$ and not on the particular choices of the α_1 and ω_1.

proof : In the representation $f = \alpha.\Delta$, α is determined by f up to an element $r \in \mathcal{R}$. But $(\alpha + r).n = \alpha.n + r.n$ and $r.n \in I$, because $n \in N_\Sigma$. The proof for ω is similar, and thus the result follows. ⊠

Note that the α and the ω - map are homomorphisms of $\mathcal{O}_\Sigma$ - modules, but in general w is only $\mathbb{C}$-linear.

Lemma (3.5) :

i) There are exact sequences

$$0 \longrightarrow P_\Sigma(\mathcal{A}) \longrightarrow N_\Sigma \xrightarrow{w} \Omega^1_\Sigma$$
$$0 \longrightarrow \int I/(f) \longrightarrow \mathcal{A} \longrightarrow P_\Sigma(\mathcal{A}) \longrightarrow 0$$

ii) There is a natural map $\varphi : P_\Sigma(\mathcal{A}) \longrightarrow \mathcal{O}/\int I$ which associates to an admissible normal vector $n \in P_\Sigma(\mathcal{A})$ a unique $\int I$ - coset such that

$$(n,g) \in \mathcal{A} \Leftrightarrow g \in \varphi(n) + \int I .$$

proof : One has: $n \in P_\Sigma(\mathcal{A}) \Leftrightarrow \exists g$ such that $(n,g) \in \mathcal{A} \Leftrightarrow \exists\ \alpha_1, \omega_1$ such that i) $g = \alpha.n + \alpha_1.\Delta$ and ii) $dg = \omega.n + \omega_1.\Delta$. When we substitute the first in the second equation, we get $d(\alpha.n) - \omega.n = -d(\alpha_1.\Delta) + \omega_1.\Delta$ Hence indeed $n \in P_\Sigma(\mathcal{A}) \Leftrightarrow w(n) = 0$, which proves the first statement. Further one has : $(0,g) \in \mathcal{A} \Leftrightarrow \exists\ \alpha_1, \omega_1$ such that i) $g = \alpha_1.\Delta$ and ii) $dg = \omega_1.\Delta$, i.e. $g \in \int I/(f)$. For the last statement, remark that by the exact sequence (2) one has that the equation $w(n) = -d(\alpha_1.\Delta) + \omega_1.\Delta$ determines the element $\alpha_1.\Delta$ modulo $\int I$. One then checks that the element $\varphi(n) = \alpha.n + \alpha_1.\Delta$ is well- defined modulo $\int I$. ⊠

To get from the space $\mathcal{A}$ of embedded admissible pairs to the space $T^1(\Sigma,X)$ of admissible first order deformations, we have to divide out by the infinitesimal automorphisms, which are generated by the vector fields $\vartheta \in \Theta$ on the ambient space:

$$T^1(\Sigma,X) = \mathcal{A} / \{(\vartheta(\Delta),\vartheta(f)) \mid \vartheta \in \Theta\} \tag{10}$$

In (3.5) $\mathcal{A}$ appears as an extension of $P_\Sigma(\mathcal{A})$ by $\int I/(f)$ and this suggests that we first divide out the automorphisms that act on $P_\Sigma(\mathcal{A})$ and then let act $\Theta(\log\Sigma) := \{\vartheta \in \Theta \mid \vartheta(I) \subset I\}$ on $\int I/(f)$.
First we have to check that indeed our maps α, ω and w descend from the space N_Σ to T^1_Σ.

Lemma (3.6) :

The α and the ω - map (and hence the w - map) descend to maps

$$\alpha : T^1_\Sigma \longrightarrow \mathcal{O}_\Sigma$$
$$\omega : T^1_\Sigma \longrightarrow \Omega^1_\Sigma$$

(which we continue to denote by the same symbols).

proof : By lemma (3.2) we have $\alpha.d\Delta + \Gamma.\Delta = 0$ for some Γ. Contracting this equation with $\vartheta \in \Theta$ then gives $\alpha.\vartheta(\Delta) \in I$. This means that the α - map descends. Let $\mathcal{L}_\vartheta$ denote the Lie-derivative with respect to ϑ. From $df = \omega.\Delta$ we then get

$$d(\vartheta(f)) = \mathcal{L}_\vartheta(df) = \mathcal{L}_\vartheta(\omega).\Delta + \omega.\vartheta(\Delta) \tag{11}$$

and as $\vartheta(f) \in I$ we indeed see $\omega.\vartheta(\Delta) = 0$ in Ω^1_Σ. ⊠

Definition/Corollary (3.7) :

Define $T^1_\Sigma(X) := \operatorname{Im}(T^1(\Sigma,X) \longrightarrow T^1_\Sigma)$. Then:

i) There are exact sequences

$$0 \longrightarrow T^1_\Sigma(X) \longrightarrow T^1_\Sigma \xrightarrow{\ w\ } \Omega^1_\Sigma$$

$$0 \longrightarrow \int I/(f, J_\Sigma(f)) \longrightarrow T^1(\Sigma,X) \longrightarrow T^1_\Sigma(X) \longrightarrow 0$$

Here $J_\Sigma(f) := \{\vartheta(f) \mid \vartheta(I) \subset I\}$.

ii) There is a natural map $\varphi : T^1_\Sigma(X) \longrightarrow \mathcal{O}/(\int I + J(f))$ which associates to an $[n] \in T^1_\Sigma(X)$ a unique coset $\varphi([n])$ such that

$$[(n,g)] \in T^1(\Sigma,X) \Leftrightarrow g \in \varphi([n]) + \int I + J(f).$$

proof : This follows immediately from (3.5) by dividing out the vector fields. Note that from (11) it follows that $\vartheta(I) \subset I \Rightarrow \vartheta(\int I) \subset \int I$. ⊠

Remark (3.8) :

R. Pellikaan ([Pe 3], pp. 19-32) studied the slightly different problem of admissible deformations of a *map* f . The space $J_\Sigma(f)$ he calls the *extended I - tangent space* to f and the number $c_{I,e}(f) = \dim_{\mathbb{C}} (\int I/J_\Sigma(f))$ he calls the *extended I - codimension.* It is clear that in our situation the space $\int I/(f, J_\Sigma(f))$ is the tangent space to the functor of admissible deformations for which Σ is kept *fixed.*

We know from § 2. that under reasonable assumptions that one has that $\mathrm{Def}(\Sigma, X) \hookrightarrow \mathrm{Def}(X)$ and hence $T^1(\Sigma, X) \hookrightarrow T^1_X$.

Proposition (3.9) :

Assume that $T^1(\Sigma, X) \hookrightarrow T^1_X$. Then one has:

i) $\int I/(f, J_\Sigma(f)) \xrightarrow{=} \int I/\int I \cap (f, J(f)) \left(\approx (\int I + J(f))/(f, J(f))\right)$

ii) $\varphi : T^1_\Sigma(X) \hookrightarrow \mathcal{O}/(\int I + J(f))$

iii) $T^1(\Sigma, X) \approx (\varphi(T^1_\Sigma(X)) + \int I + J(f))/(f, J(f)) \subset T^1_X$.

proof : i) $\int I/(f, J_\Sigma(f))$ is a subspace of $T^1(\Sigma, X)$. If this is to inject in T^1_X , then the map in i) is injective. As it is clearly surjective, it is an isomorphism. (c.f. [Pe 3], prop. 5.3). The injectivity in ii) expresses just the fact that if $[(n, g)] \in T^1(\Sigma, X)$, and $T^1(\Sigma, X) \hookrightarrow T^1_X$, then the deformation $[n]$ of Σ is essentially determined by g. Statement iii) then follows from i) and ii). ⊠

The following proposition gives the dependence of the w - map on our function $f \in \int I$.

Proposition (3.10) :

i) There is a $\mathbb{C}$- bilinear map

$$W : \int I/I^2 \times T^1_\Sigma \longrightarrow \Omega^1_\Sigma$$
$$(f \;,\; n) \longmapsto w_f(n)$$

where $w_f(n) = d(\alpha.n) - \omega.n$, $f = \alpha.\Delta$ and $df = \omega.\Delta$.

ii) If $f \in I^2$ (so $f = (h.\Delta).\Delta$ for some symmetric matrix h) then we have $P_\Sigma(\mathcal{A}) = N_\Sigma$ and the induced map $\varphi : N_\Sigma \longrightarrow \mathcal{O}/\int I$ is given by $n \longmapsto 2.(h.\Delta).n$

proof : The above map is clearly linear in f. We show that if $f \in I^2$, then w_f is the zero map. But if $f = h.\Delta.\Delta$ for some matrix h, then we can take $\alpha = h.\Delta$ and $\omega = d(h).\Delta + 2.h.d\Delta$, so $w_f(n) = 0$ in Ω^1_Σ. ⊠

Above we considered the space $T^1(\Sigma,X)$ for a general Σ. In the situation we are most interested in there is an important simplification.

Lemma (3.11) :

Assume that Σ is *reduced.* Then the α - map

$$\alpha : T^1_\Sigma \longrightarrow \mathcal{O}_\Sigma$$

is the zero map.

proof : If Σ is reduced, then T^1_Σ is a torsion $\mathcal{O}_\Sigma$ - module. As $\mathcal{O}_\Sigma$ is torsion free, α has to be the zero map. ⊠

The α - map being the zero map has the effect of making all maps we encountered not only ($\mathbb{C}$-) linear, but even *module homomorphisms.*

Proposition (3.12) :

If the α - map is the zero map one has:

i) $(n,g) \in \mathcal{A} \Rightarrow g \in I$.

ii) $w : N_\Sigma \longrightarrow \Omega^1_\Sigma$ is $\mathcal{O}_\Sigma$ - linear.

iii) $\varphi : P_\Sigma(\mathcal{A}) \longrightarrow I/\int I$ is $\mathcal{O}_\Sigma$ - linear.

iv) $\mathcal{A}$ and $T^1(\Sigma,X)$ are $\mathcal{O}_X$ - modules.

v) $P_\Sigma(\mathcal{A})$ and $T^1_\Sigma(X)$ are $\mathcal{O}_\Sigma$ - modules.

We omit the easy and straight forward proof.

Remark (3.13) :

In general there is an exact sequence of the form

$$0 \longrightarrow N_{\Sigma,X} \longrightarrow N_{\Sigma} \xrightarrow{\alpha} \mathcal{O}_{\Sigma}$$

where $N_{\Sigma,X} = \mathrm{Hom}_X(I \otimes \mathcal{O}_X, \mathcal{O}_{\Sigma})$ is the normal bundle of Σ inside X. So the equality of these two normal bundles is in fact equivalent to the α - map being the zero map. In § 4. we will encounter another interpretation of the condition that α is the zero map.

Corollary (3.14) :

If Σ is reduced, then the map $w: T^1_{\Sigma} \longrightarrow \Omega^1_{\Sigma}$ in fact lands in the torsion sub-sheaf $\mathrm{Tors}(\Omega^1_{\Sigma})$, i.e. $w : T^1_{\Sigma} \longrightarrow \mathrm{Tors}(\Omega^1_{\Sigma})$.

proof : This is clear, because w is $\mathcal{O}_{\Sigma}$ - linear and T^1_{Σ} is torsion. ⊠

We summarize the above discussion in a theorem.

Theorem (3.15) :

Let Σ be defined by an ideal I and let $f \in \int I$. Assume that:

i) Σ is reduced and Cohen - Macaulay.

ii) $\dim(\Sigma) \geq 1$.

iii) $\dim(I/(f,J(f)) < \infty$.

Then one has:

$$T^1(\Sigma,X) \approx P_X(\mathcal{A})/(f,J(f)) \subset I/(f,J(f)) \subset T^1_X .$$

where $P_X(\mathcal{A})$ is the *ideal* $\left(\varphi(T^1_{\Sigma}(X)) + \int I + J(f)\right)$

and $T^1_{\Sigma}(X) = \mathrm{Ker}(\, w : T^1_{\Sigma} \longrightarrow \mathrm{Tors}(\Omega^1_{\Sigma}))$.

proof : Use (2.4), (3.9), (3.11), (3.12) and (3.14). ⊠

We conclude this part with some simple examples.

Examples (3.16) :

1) $f = xyz \in \mathbb{C}\{x,y,z\}$, $I = (y.z,\, z.x,\, x.y)$. Because $I/J(f) = 0$ we have $T^1(\Sigma,X) = 0$. In this example one has $w : T^1_{\Sigma} \xrightarrow{\approx} \mathrm{Tors}(\Omega^1_{\Sigma})$ and $T^1_{\Sigma}(X) = 0$.

2) $f = xy^2 \in \mathbb{C}\{x,y\}$, $I = (y)$.

Then $P_X(\mathcal{A}) = (xy, y^2) = J(f)$, hence $T^1(\Sigma, X) = 0$.

3) $f = x^2y^2 + y^4$, $I = (y)$.

Then $P_X(\mathcal{A}) = (x^2y, y^2)$ and $J(f) = (2x^2y + 4y^3, xy^2)$

Hence $T^1(\Sigma, X)$ is 2 - dimensional.

4) $f = (yz)^2 + (zx)^2 + (xy)^2$; $I = (yz, zx, xy)$.

Because $f \in I^2$, $w : N_\Sigma \longrightarrow \Omega^1_\Sigma$ is the zero map. Hence $P_\Sigma(\mathcal{A}) = N_\Sigma$ and is generated by the following vectors:

$(y,0,0)$, $(z,0,0)$, $(0,x,0)$, $(0,z,0)$, $(0,0,x)$, $(0,0,y)$.

A calculation shows that:

$P_X(\mathcal{A}) = (y^2z, yz^2, z^2x, zx^2, x^2y, xy^2, xyz)$ and

$(f, J(f)) = (xy^2+xz^2, x^2z+y^2z, x^2y+z^2y)$.

Hence dim $T^1(\Sigma,X) = 7$, with as basis:

$\{3xyz, 2(y^2z+yz^2), 2(x^2z+xz^2), 2(x^2y+xy^2), 2x^2yz, 2xy^2z, 2xyz^2\}$.

Conjecture (3.17) :

Let X be a germ of a weakly normal surface singularity in $\mathbb{C}^3$ with singular locus Σ. Then:

$$T^1(\Sigma,X) = 0 \Leftrightarrow X \approx A_\infty \quad : f = y^2 + z^2 \quad \text{or}$$
$$\approx D_\infty \quad : f = x.y^2 + z^2 \quad \text{or}$$
$$\approx T_{\infty,\infty,\infty} : f = xyz$$

Remark (3.18) :

We will see in § 4 the reason why one in general can *not* expect that for every $\Sigma \hookrightarrow X$ there is an admissible deformation such that on the general fibre the space X_η has only singularities as in (3.17). Such a deformation we call a *disentanglement* (see (5.6). In fact, if $\mathcal{E} \subset \mathbb{P}^2$ is a curve with ordinary nodes which is birational to a non-hyperelliptic curve of sufficiently high degree, then $X = \mathrm{Cone}(\mathcal{E})$, $\Sigma = \mathrm{Sing}(X)_{red}$ is an example of a pair $\Sigma \hookrightarrow X$ which has essentially only 'equisingular' admissible deformations (c.f. [Mu]). In particular, it can not have a disentanglement.

B. Obstruction Theory and Semi - Universal Deformation.

In this part we consider the problem of extending a given admissible deformation over a space to a slightly bigger space. We will show that in our situation the theory is completely analogous to the ordinary deformation theory. We also outline the steps that lead to the construction of the base space of the semi - universal admissible deformation. As an application we prove a theorem stating that this base space only depends, up to a smooth factor, on the class of $f \in \int I$ modulo I^2.

Consider a 'small surjection' of rings $S' \longrightarrow S$, i.e. suppose we are given an exact sequence of the form:

$$0 \longrightarrow V \longrightarrow S' \longrightarrow S \longrightarrow 0 \qquad (12)$$

where V is an ideal in S' with the property $V.m_{S'} = 0$. In this situation V becomes an S - module, in fact a module over $k = S/m_S$. We study the map $\rho : Def(\Sigma, X)(S') \longrightarrow Def(\Sigma, X)(S)$. The questions that arize are:

* What is the image of ρ, i.e. which deformations over S can be extended to deformations over S' ?
* What are the fibres of ρ, i.e. in how many different ways can one extend a given deformation over S to one over S' ?

Given a deformation $\Sigma_S \hookrightarrow X_S \in Def(\Sigma, X)(S)$, we split up the above problem into three steps.

1. Try to lift Σ_S to $\Sigma_{S'}$ over S'. This is an ordinary deformation problem for Σ.
2. Given a lift $\Sigma_{S'}$ of Σ_S, try to find $X_{S'}$ as to make an admissible deformation with $\Sigma_{S'}$.
3. Vary the choice of $\Sigma_{S'}$ in 2. to get from a given $\Sigma_S \hookrightarrow X_S \in Def(\Sigma, X)(S)$ to an element $\Sigma_{S'} \hookrightarrow X_{S'} \in Def(\Sigma, X)(S')$.

About step 1. we quote the following theorem.

Theorem (3.19) :

Given a deformation $\Sigma_S \in \mathrm{Def}(\Sigma)(S)$ and a small surjection as in (12), then there is an element

$$\mathrm{Ob}(\Sigma_S) \in T^2_\Sigma \otimes V$$

(where $T^2_\Sigma := \mathrm{Hom}(\mathcal{R}/\mathcal{R}_0, \mathcal{O}_\Sigma)/\mathrm{Hom}(\mathcal{F}, \mathcal{O}_\Sigma)$, where $\mathcal{R}$, $\mathcal{F}$ as in (6) and $\mathcal{R}_0$ is the module of 'Koszul relations') with the following properties:

i) Σ_S extends to an $\Sigma_{S'}$ $\Leftrightarrow$ $\mathrm{Ob}(\Sigma_S) = 0$.

ii) If $\mathrm{Ob}(\Sigma_S) = 0$, then the possible choices for $\Sigma_{S'}$ form a principal homogeneous space over $T^1_\Sigma \otimes V$.

proof : Well - known, see [Schl 2], pp. 149-150. ⊠

(We use $\otimes$ for tensor products over the ring and $\otimes$ for tensor products over the ground field k.)

Given an admissible deformation $\Sigma_S \hookrightarrow X_S \in \mathrm{Def}(\Sigma, X)(S)$ we thus get a *first obstruction* $\mathrm{Ob}(\Sigma_S) \in T^2_\Sigma \otimes V$. If this obstruction vanishes we choose an $\Sigma_{S'}$ and go on with step 2. for which we have the following.

Proposition (3.20) :

Given an admissible deformation $\xi_S = \left(\Sigma_S \hookrightarrow X_S\right) \in \mathrm{Def}(\Sigma, X)(S)$ and a lift $\Sigma_{S'}$ of Σ_S, then there exists an element

$$\mathrm{Ob}(\xi_S, \Sigma_{S'}) \in \Omega^1_\Sigma \otimes V$$

with the following properties:

i) There is an $X_{S'}$ such that $\left(\Sigma_{S'} \hookrightarrow X_{S'}\right) \in \mathrm{Def}(\Sigma, X)(S')$ $\Leftrightarrow$ $\mathrm{Ob}(\xi_S, \Sigma_{S'}) = 0$.

ii) If $\mathrm{Ob}(\xi_S, \Sigma_{S'}) = 0$, then the possible choices for $X_{S'}$ form a principal homogeneous space over $\int I/(f, J_\Sigma(f)) \otimes V$.

proof : Let $P_S = \mathcal{O}_{\mathbb{C}^{n+1}\times S}$ the local ring of the ambient space over S and let $P_{S'}$ be defined analogous. Then an element ξ_S is represented by α_S and Γ_S such that (see (3.2)) :

$$\alpha_S \cdot \Delta_S = f_S$$

$$\alpha_S \cdot d\Delta_S + \Gamma_S \cdot \Delta_S = 0$$

Here Σ_S is defined by the ideal $I_S = (\Delta_S)$ and X_S by f_S.
Note that there is an exact sequence of Kähler differentials:

$$0 \longrightarrow V \otimes \Omega^1_{P_{S'}/S'} \longrightarrow \Omega^1_{P_{S'}/S'} \longrightarrow \Omega^1_{P_S/S} \longrightarrow 0$$

and an isomorphism $V \otimes \Omega^1_{P_{S'}/S'} \xrightarrow{\approx} V \otimes \Omega_P$, $\Omega_P = \Omega^1_{\mathbb{C}^{n+1}}$.

Now given is a lift $\Sigma_{S'}$ of Σ_S , i.e. we have $\Delta_{S'}$. Take *any* lift of α_S to $\alpha_{S'}$ and Γ_S to $\Gamma_{S'}$ and consider the element

$$w_{S'} = \alpha_{S'} . d\Delta_{S'} + \Gamma_{S'} . \Delta_{S'} \in \Omega^1_{P_{S'}/S'}$$

Because over S $w_{S'}$ is the zero - form, we see that via the above isomorphism we can consider $w_{S'}$ as an element of $V \otimes \Omega^1_P$.

Claim : The class of $w_{S'}$ in $\Omega^1_\Sigma \otimes V$ is independent of the choice of $\alpha_{S'}$ and $\Gamma_{S'}$. In fact, the differences of two choices of $\alpha_{S'}$ and $\Gamma_{S'}$ are of the form $v_1 \otimes \beta_{S'}$ and $v_2 \otimes \gamma_{S'}$, $v_i \in V$. Hence the difference of the $w_{S'}$ is of the form $v_1 \otimes \beta_{S'}.d\Delta_{S'} + v_2 \otimes \gamma_{S'}.\Delta_{S'} = v_1 \otimes \beta.d\Delta + v_2 \otimes \gamma\Delta$, because $V.m_{S'} = 0$. Consequentely, the class of $w_{S'}$ in $\Omega^1_\Sigma \otimes V$ is only dependent on $\xi_S = \left(\Sigma_S \hookrightarrow X_S\right)$ and $\Sigma_{S'}$. We put:

$$Ob(\xi_S, \Sigma_{S'}) = [w_{S'}] \in \Omega^1_\Sigma \otimes V .$$

From the exact sequence (2)

$$0 \longrightarrow I/\int I \xrightarrow{d} \Omega^1_P \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0$$

we see that extension to S' is possible if and only $[w_{S'}] = 0$ and that then the choice $f_{S'} = \alpha_{S'} . \Delta_{S'}$ is determined modulo $\int I \otimes V$. Dividing out the isomorphisms for $\Sigma_{S'} \hookrightarrow X_{S'}$ then leads to a principal homogeneous space over $\int I/(f, J_\Sigma(f)) \otimes V$, as in (3.7). ⊠

Finally, the result about step 3. is the following.

Theorem (3.21) :

Let $\xi_S = \left(\Sigma_S \hookrightarrow X_S\right) \in \mathrm{Def}(\Sigma,X)(S)$ be an admissible deformation over S. Then one has the following:

i) There is a *first obstruction* $\mathrm{Ob}(\Sigma_S) \in T^2_\Sigma \otimes V$.

ii) If $\mathrm{Ob}(\Sigma_S) = 0$, then there is a *second obstruction* $\mathrm{Ob}(\xi_S) \in T^2(\Sigma,X) \otimes V$.

iii) ξ_S lifts to an element $\xi_{S'} = \left(\Sigma_{S'} \hookrightarrow X_{S'}\right) \in \mathrm{Def}(\Sigma,X)(S') \Leftrightarrow$ $\mathrm{Ob}(\Sigma_S) = 0$ and $\mathrm{Ob}(\xi_S) = 0$.

iv) The possible choices for $\xi_{S'}$ in iii) form a principal homogeneous space over $T^1(\Sigma,X) \otimes V$.

Here $T^2(\Sigma,X) := \mathrm{Coker}(w : T^1_\Sigma \longrightarrow \Omega^1_\Sigma)$ is the *obstruction space* of our problem.

proof : This readily follows from (3.19) and (3.20). We only have to study the behaviour of the class $\mathrm{Ob}(\xi_S, \Sigma_{S'})$ when we vary the lift $\Sigma_{S'}$ of Σ_S. Two different choices of $\Delta_{S'}$ differ by an element $n \in N_\Sigma \otimes V$, so $\mathrm{Ob}(\xi_S, \Sigma_{S'})$ descends to a class $\mathrm{Ob}(\xi_S) \in T^2(\Sigma,X) \otimes V$ as defined above. (We omit some further details.) ⊠

Remark (3.22) :

If the α - map of (3.3) is the zero - map (c.f. (3.11)), then $T^2(\Sigma,X)$ is not only a vector space, but also an $\mathcal{O}_\Sigma$ - module. In fact, in that case $w : T^1_\Sigma \longrightarrow \Omega^1_\Sigma$ is $\mathcal{O}_\Sigma$ - linear and $T^2(\Sigma,X) = \mathrm{Coker}(w)$.

When we assume that Σ is reduced then T^1_Σ is a torsion $\mathcal{O}_\Sigma$ - module. In that case it can be seen that the obstruction $\mathrm{Ob}(\xi_S)$ in fact lands in the $\mathcal{O}_\Sigma$ - module $\mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma)$. In the case that Σ has an isolated singular point, this is a finite dimensional vector space.

We now apply Schlessingers method (see [Schl1], pp. 213-215) to construct the *hull* of the functor $\mathrm{Def}(\Sigma, X)$. For reasons of simplicity we will assume that Σ is a reduced space with an isolated singular point (see (3.22)) and that $T^2_\Sigma = 0$. (For example, Σ could be the germ of a space curve in $\mathbb{C}^3$.)

Procedure (3.23) :

* Take a basis $t_1, t_2, \ldots, t_\tau$ of the dual space of $T^1(\Sigma, X)$, and a basis $v_1, v_2, \ldots, v_\sigma$ for the dual space of $\mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma)$.
* Consider the ring $U := \mathbb{C}[[t_1, t_2, \ldots, t_\tau]]$. We are going to define a sequence of ideals $J_2 \supset J_3 \supset \ldots \supset J_q \supset \ldots$ in U, together with elements $\xi_q \in \mathrm{Def}(\Sigma, X)(U_q)$, where $U_q = U/J_q$.
* The ideals J_q and the deformations ξ_q are defined inductively. One puts $J_2 = m^2$ and constructs the universal first order family ξ_2. In term of equations ξ_2 is given by $\alpha_i^{(1)}$, $\Gamma_i^{(1)}$ and $n_i^{(1)}$ such that
  $\left(f + \sum t_i . g^{(1)}\right) = \left(\alpha + \sum t_i . \alpha_i^{(1)}\right) . \left(\Delta + \sum t_i . n_i^{(1)}\right)$ and
  $0 = \left(\alpha + \sum t_i . \alpha_i^{(1)}\right) . d\left(\Delta + \sum t_i . n_i^{(1)}\right) + \left(\Gamma + \sum t_i . \Gamma_i^{(1)}\right) . \left(\Delta + \sum t_i . n_i^{(1)}\right)$
  where both equations are mod m^2.

  Now assume we have constructed $\xi_q \in \mathrm{Def}(\Sigma, X)(U_q)$. We look for an ideal J_{q+1}, $m . J_q \subset J_{q+1} \subset J_q$ which is *minimal* with respect to the property that ξ_q extends to an $\xi_{q+1} \in \mathrm{Def}(\Sigma, X)(U_{q+1})$.
* A way to obtain such a J_{q+1} is as follows. Consider the small surjection

  $$0 \longrightarrow J_q / m . J_q \longrightarrow U / m . J_q \longrightarrow U_q \longrightarrow 0$$

  We get an obstruction $\mathrm{Ob}(\xi_q) \in \mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma) \otimes \left(J_q / m . J_q\right)$. Applying the elements $v_1, v_2, \ldots, v_\sigma$ to $\mathrm{Ob}(\xi_q)$ we get elements $V_1, V_2, \ldots, V_\sigma$ in $J_q / m . J_q$. It is now clear that one can take $J_{q+1} = (V_1, V_2, \ldots, V_\sigma, m . J_q)$. Now one *chooses any* ξ_{q+1} *lifting the element* ξ_q and then considers $\mathrm{Ob}(\xi_{q+1})$ etc.

One easily sees by induction from the above construction that the ideal J_q is generated by σ elements $V_1, V_2, \ldots, V_\sigma$ and m^q. We call the elements $V_i \bmod m^q$, $i = 1, 2, \ldots, \sigma$, *the equations of the base space of the semi - universal admissible deformation to order q-1* or simply *the equations of the base space*. It should be stressed that these equations do not only depend on the chosen bases t_i and v_j, but also on the chosen lifts ξ_q. In practice this is of crucial importance: good choices can simplify the equations a lot, whereas bad choices make the computations into a nightmare.

Example (3.24) :

This example is a continuation of example (3.16) 4). As the map $w : N_\Sigma \longrightarrow \Omega^1_\Sigma$ is the zero map, we have that the obstruction space is $\mathrm{Tors}(\Omega^1_\Sigma)$. One easily computes that the dimension of this space is three and has as a basis:

$$w_1 = x.d(y+z);\ w_2 = y.d(z+x);\ w_3 = z.d(x+y)\ .$$

The semi-universal deformation to first order is described by the following data:

a) the deformed curve: $\Delta_1 = \Delta + \sum_{i=1}^{3} t_i.n_i$, where

$\Delta = (yz, zx, xy);\ n_1 = (y+z, 0, 0);\ n_2 = (0, z+x, 0);\ n_3 = (0, 0, x+y)$.

b) the deformed α's: $\alpha_1 = h_1.\Delta_1 + t_0.(x,y,z)$, where

$$h_1 = \begin{pmatrix} 1 & t_4 & t_5 \\ t_4 & 1 & t_6 \\ t_5 & t_6 & 1 \end{pmatrix}$$

c) the deformed f is $f_1 = \alpha_1.\Delta_1 \pmod{m^2}$.

d) the deformed Γ's : $\Gamma_1 = - h_1.d\Delta_1 - 2t_0.(dx, dy, dz)$.

The curve is not obstructed, and a lift to second order is given by:

$$\Delta_2 = \Delta_1 + \sum_{i\neq j}^{3} t_i.t_j(1,1,1)$$

The obstruction element $\alpha_1 d\Delta_2 + \Gamma_1.\Delta_2$ in $\mathrm{Tors}(\Omega^1_\Sigma)\otimes(m^2/m^3)$ is

$$t_0.(x,y,z).d\,(\textstyle\sum_{i=1}^{3} t_i.n_i) - 2t_0.(dx, dy, dz).(\sum_{i=1}^{3} t_i.n_i)$$

Because in Ω^1_Σ we have the relation $(y+z)dx + x\,d(y+z) = d(xy+xz) = 0$, we can rewrite this expression as:

$$3t_0t_1 \cdot w_1 + 3t_0t_2 \cdot w_2 + 3t_0t_3 \cdot w_3 \in \mathrm{Tors}(\Omega^1_\Sigma) \otimes (m^2/m^3) \ .$$

Hence the equations for the base to second order are given by:

$$t_0t_1 = 0 \ ; \qquad t_0t_2 = 0 \ ; \qquad t_0t_3 = 0 \ .$$

A lift of f to second order is given by $f_2 = (h_1 \cdot \Delta_2).\Delta_2 + t_0\,(x,y,z).\Delta_2$ and one can check that this family, as it stands, defines an admissible family to every order.

Suppose that we are given admissible diagrams $\Sigma \hookrightarrow X^{(1)}$ and $\Sigma \hookrightarrow X^{(2)}$, where $X^{(i)}$ is described by $f^{(i)} \in \int I$. When $f^{(1)} - f^{(2)} \in I^2$, then (3.10) implies that the first order obstruction map for $X^{(1)}$ and $X^{(2)}$ are 'the same'. In fact, the following propositions show that when $f^{(1)} - f^{(2)} \in I^2$ something much stronger is true: $\mathrm{Def}(\Sigma, X^{(1)})$ and $\mathrm{Def}(\Sigma, X^{(2)})$ are the same up to a smooth factor.

Proposition (3.25) :

Let $\xi_S^{(i)} = (\Sigma_S \hookrightarrow X_S^{(i)}) \in \mathrm{Def}(\Sigma, X^{(i)})$, $i = 1,2$ and let be given a small surjection as in (12).

Let $X_S^{(i)}$ be defined by $f_S^{(i)} \in \int I_S$. If $f_S^{(1)} - f_S^{(2)} \in I_S^2$ then we have:

i) $T^2(\Sigma, X^{(1)}) = T^2(\Sigma, X^{(2)})$.

ii) $\mathrm{Ob}(\xi_S^{(1)}) = \mathrm{Ob}(\xi_S^{(2)})$.

iii) Let $S' \longrightarrow S$ any surjection. Then $\xi_S^{(1)}$ can be lifted to S' if and only if $\xi_S^{(2)}$ can be lifted to S'. If this is the case, we can do this in such a way that $f_{S'}^{(1)} - f_{S'}^{(2)} \in I_{S'}^2$.

proof : We have $T^2(\Sigma, X^{(1)}) = \mathrm{Coker}(w^{(1)} : T^1_\Sigma \longrightarrow \Omega^1_\Sigma)$, and by (3.10) we have $w^{(1)} = w^{(2)}$, hence we get i). For ii) we assume that Σ_S is lifted to $\Sigma_{S'}$. As $f_S^{(1)} - f_S^{(2)} = h_S \cdot \Delta_S \cdot \Delta_S$ for some matrix h_S , we can take $\alpha_S^{(1)} - \alpha_S^{(2)} = h_S \cdot \Delta_S$ and $\Gamma_S^{(1)} - \Gamma_S^{(2)} = h_S \cdot d\Delta_S$. Now lift h_S to a matrix $h_{S'}$ and $\alpha_S^{(1)}$, $\Gamma_S^{(1)}$ over S'. Define then the lifts for $\alpha_S^{(2)}$, $\Gamma_S^{(2)}$ by requiring the above relations to hold over S'. Then one has $w_{S'}^{(1)} = w_{S'}^{(2)}$, so $\mathrm{Ob}(\xi_S^{(1)}) = \mathrm{Ob}(\xi_S^{(2)})$. Statement iii) can be deduced from ii) by factoring the surjection in a sequence of small surjections. The indicated choices above lead to $f_{S'}^{(1)} - f_{S'}^{(2)} \in I_{S'}^2$. ⊠

Definition (3.26) :

Two deformations $\xi_S^{(i)}=(\Sigma_S, X_S^{(i)}) \in Def(\Sigma,X)$ $(i=1,2)$ are called *I^2-equivalent* if there are equations $f_S^{(i)}$ for $X_S^{(i)}$ such that $f_S^{(1)} - f_S^{(2)} \in I_S^2$, where I_S is the ideal of Σ_S. We denote this equivalence relation by $\sim$.
The functor

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{Set} \\ S & \longmapsto & Def(\Sigma,X)/\sim \end{array}$$

is denoted by $M(\Sigma,X)$ and is called the functor of *admissible deformations modulo I^2*.

Proposition (3.27) :

Let $\Sigma \hookrightarrow X$ be an admissible diagram. Then one has:

i) The natural transformation $Def(\Sigma,X) \longrightarrow M(\Sigma,X)$ is smooth.

ii) $M(\Sigma,X)$ is a semi-homogeneous functor.

iii) If $X^{(1)}$ is I^2- equivalent to $X^{(2)}$, then there is a natural equivalence of functors $M(\Sigma,X^{(1)}) \approx M(\Sigma,X^{(2)})$.

iv) The space $M^1(\Sigma,X) := M(\Sigma,X)(k[\varepsilon]/(\varepsilon^2))$ fits in an exact sequence:

$$0 \longrightarrow \int I/\big((f,J_\Sigma(f)) + I^2\big) \longrightarrow M^1(\Sigma,X) \longrightarrow T^1_\Sigma(X) \longrightarrow 0$$

Sketch of proof : Statement i) follows from (3.25). Statement ii) can be proved by showing that $\sim$ is an 'admissible' equivalence relation in the sense of [Bu], p23. (The word *admissible* here should not confuse the reader.) For this one uses i). Statement iii) is essentially trivial: the equivalence of functors is defined in the obvious way on representatives. The fact that $\sim$ is really an equivalence relation then shows that it is well-defined. Statement iv) is proved along the lines of (3.7)i). Note that by (3.10)the space $T^1_\Sigma(X)$ depends indeed only on the class of f modulo I^2. It is easy to see that the first space in the exact sequence iv) depends only on $[f] \in \int I/I^2$. ⊠

Corollary (3.28) :

If $\int I/(I^2 + f^{(1)}) = \int I/(I^2 + f^{(2)})$, then the base space of $\Sigma \hookrightarrow X^{(1)}$ and $\Sigma \hookrightarrow X^{(2)}$ are the same up to a smooth factor.

From now on we consider an admissible diagram $\Sigma \hookrightarrow X$ with Σ *reduced*. In B. we have seen that the obstruction space for the functor $\mathrm{Def}(\Sigma, X)$ is, apart from T^2_Σ, the $\mathcal{O}_\Sigma$ - module $\mathrm{Tors}(\Omega^1_\Sigma)/w(T^1_\Sigma)$. This space however seems to be too big in general. For example, when Σ is a reduced complete intersection, then $\int I = I^2$ and hence we know that all obstructions vanish. But $\mathrm{Tors}(\Omega^1_\Sigma) \neq 0$ (unless Σ is smooth) and $w(T^1_\Sigma) = 0$, so the obstruction space is never zero. In fact, it is a long standing conjecture that for a singular curve germ one always has $\mathrm{Tors}(\Omega^1_\Sigma) \neq 0$. (Berger's conjecture, see [Be].) It turns out that in the case that $T^2_\Sigma = 0$ there is a subspace of $\mathrm{Tors}(\Omega^1_\Sigma)$ which receives all the obstructions. Let us first describe this subspace, which we call N^*/I. Recall the exact sequence (2) of $\mathcal{O}_\Sigma$ - modules:

$$0 \longrightarrow I/\textstyle\int I \xrightarrow{\ d\ } \Omega^1 \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0$$

Now take the double dual of the map d. As $\mathrm{Hom}_\Sigma(I/\int I, \mathcal{O}_\Sigma) = N_\Sigma$ and $\Omega^1 \otimes \mathcal{O}_\Sigma$ is $\mathcal{O}_\Sigma$ - free, we get:

$$0 \longrightarrow N_\Sigma^* \xrightarrow{\ d^{**}\ } (\Omega^1 \otimes \mathcal{O}_\Sigma)^{**} \approx \Omega^1 \otimes \mathcal{O}_\Sigma$$

where $N_\Sigma^* = \mathrm{Hom}_\Sigma(N_\Sigma, \mathcal{O}_\Sigma)$ is the *dual* of the normal bundle. Further, there is the double duality inclusion $I/\int I \hookrightarrow N_\Sigma^*$ and hence we get an inclusion:

$$N^*/I := N_\Sigma^*/(I/\textstyle\int I) \hookrightarrow \mathrm{Tors}(\Omega^1_\Sigma) \subset \Omega^1_\Sigma \qquad (13)$$

(N^*/I is a torsion $\mathcal{O}_\Sigma$ - module, so it lands in $\mathrm{Tors}(\Omega^1_\Sigma)$.)
Note that if Σ is a complete intersection, then I/I^2 is a free $\mathcal{O}_\Sigma$ - module, and hence $N^*/I = 0$.

Conjecture (3.29) :

$N^*/I = 0$ if and only if Σ is a complete intersection.

We have to admit however that we do not have overwhelming evidence for the truth of this conjecture. In practice it is hard to compute the space N^*/I. If Σ is Cohen-Macaulay of codimension 2, one can prove that $N^*/I \approx \mathcal{E}xt^2_\Sigma(\omega_\Sigma, \mathcal{O}_\Sigma)$, but this does not seem to be of great help to settle the conjecture even for space curves. Note that conjecture (3.29) implies Berger's conjecture, because for a complete intersection curve singularity Berger's conjecture is known to be true.

Lemma (3.30) :

Via the canonical injection $I/\int I \hookrightarrow N_\Sigma^*$ the element $\beta.\Delta \in I$ is send to the homomorphism $\beta: N_\Sigma \longrightarrow \mathcal{O}_\Sigma;\ n \longmapsto \beta.n$.

proof : Disentangle the double duality definition. ⊠

From the above we see that for an $f \in I$ one has (c.f. (3.11):

$$f \in \int I \quad \Leftrightarrow \quad \text{the } \alpha \text{ - map of } f \text{ is the zero map.}$$

As the above statement can be made to work over any base S, we get an alternative way to express the condition that $\Sigma_S \hookrightarrow X_S$ is an *admissible* deformation. First we define the ***relative primitive ideal*** $\int I_S$ as follows:

$$f_S \in \int I_S \Leftrightarrow \Sigma_S \subset \mathcal{C}_{X_S/S} \left(\Leftrightarrow (f_S, \partial_i f_S) \subset I_S \right).$$

In other words, there is an exact sequence

$$0 \longrightarrow I_S/\int I_S \xrightarrow{d} \Omega^1_{P_S/S} \otimes \mathcal{O}_{\Sigma_S} \longrightarrow \Omega^1_{\Sigma_S/S} \longrightarrow 0 \qquad (14)$$

which is the relative version of (2).

Theorem (3.31) :

Let Σ be a reduced space and let $\Sigma_S \longrightarrow S$ be a deformation of Σ over S. Let I_S be the ideal of Σ_S. Then there is an inclusion

$$I_S/\int I_S \hookrightarrow N_{\Sigma_S}^*$$

and hence an equivalence

$$\Sigma_S \hookrightarrow X_S \text{ admissible } \left(\Leftrightarrow f_S \in \int I_S\right) \Leftrightarrow$$
$$\text{the } \alpha \text{ - map } \alpha_S : N_{\Sigma_S} \longrightarrow \mathcal{O}_{\Sigma_S} \text{ is the zero map.}$$

proof : The (almost) dual of (14) is the exact sequence

$$0 \longrightarrow \Theta_{\Sigma_S} \longrightarrow \Theta_{P_S/S} \otimes \mathcal{O}_{\Sigma_S} \longrightarrow N_{\Sigma_S} \longrightarrow T^1_{\Sigma_S/S} \longrightarrow 0$$

where the group at the right hand side can be interpreted as the first order deformations of the map $\Sigma_S \longrightarrow \mathrm{Spec}(S)$. The dual of this sequence starts with the exact segment:

$$0 \longrightarrow \mathrm{Hom}_{\Sigma_S}(T^1_{\Sigma_S/S}, \mathcal{O}_{\Sigma_S}) \longrightarrow N^*_{\Sigma_S} \longrightarrow \Omega^1_{P_S/S} \otimes \mathcal{O}_{\Sigma_S}$$

We claim that the group at the left hand side is actually zero.
Using (0.3) we can conclude:

$$\mathrm{Hom}_{\Sigma}(T^1_{\Sigma_S/S} \otimes \mathcal{O}_{\Sigma}, \mathcal{O}_{\Sigma}) = 0 \quad \Rightarrow \quad \mathrm{Hom}_{\Sigma_S}(T^1_{\Sigma_S/S}, \mathcal{O}_{\Sigma_S}) = 0.$$

But as Σ is reduced by assumption, and $T^1_{\Sigma_S/S} \otimes \mathcal{O}_{\Sigma}$ is a torsion $\mathcal{O}_{\Sigma}$ module, this first Hom is indeed zero. ⊠

This alternative way to express admissibility of $\Sigma_S \hookrightarrow X_S$ also leads to an alternative obstruction theory. For this to work we need an extra condition on Σ.

Lemma (3.32) :

Assume that $T^2_{\Sigma} = 0$.

i) The normal bundle is compatible with restriction, i.e. if $S' \longrightarrow S$ is a surjection of rings and $\Sigma_{S'} \longrightarrow \mathrm{Spec}(S')$ is a deformation, then one has $N_{\Sigma_{S'}} \otimes S = N_{\Sigma_S}$.

ii) The exact sequence (12) gives rise to an exact sequence

$$0 \longrightarrow V \otimes N_{\Sigma_{S'}} \longrightarrow N_{\Sigma_{S'}} \longrightarrow N_{\Sigma_S} \longrightarrow 0$$

and an isomorphism $V \otimes N_{\Sigma_{S'}} \approx V \otimes N_{\Sigma}$ (i.e. N_{Σ_S} is flat).

Outline of proof : It is enough to show this for small extensions. N_{Σ_S} can be interpreted as the space of (embedded) deformations of the *map* $\Sigma \longrightarrow \mathrm{Spec}(S)$ over $k[\varepsilon]/(\varepsilon\,)$. With this interpretation, statement i) is equivalent to the extendability of a family over $S' \times S[\varepsilon]/(\varepsilon^2)$ to a family over $S'[\varepsilon]/(\varepsilon^2)$. This is certainly implied by the condition $T^2_{\Sigma}=0$. Statement ii) now follows from (0.1) (take $i=0, R=P, M=I_S, N=\mathcal{O}_{\Sigma_S}$.)

⊠

Let $\xi_S = \left(\Sigma_S \hookrightarrow X_S\right) \in Def(\Sigma,X)(S)$ be an admissible deformation over S and $\Sigma_{S'} \in Def(\Sigma)(S')$ a deformation of Σ over S', lifting Σ_S . We will construct an element $ob(\xi_S,\Sigma_{S'}) \in N/I \otimes V$, which maps to the element $Ob(\xi_S, \Sigma_{S'}) \in \Omega^1_\Sigma \otimes V$ of (3.20) via the map (13).
The construction is as follows:

* Over S we know that the α - map $\alpha_S : N_{\Sigma_S} \longrightarrow \mathcal{O}_{\Sigma_S}$ is the zero map by (3.29). Hence, for all $m_S \in N_{\Sigma_S}$ there is a $\gamma_S = \gamma_S(m_S)$ such that $\alpha_S . m_S + \gamma_S . \Delta_S = 0$.
* Now a lift of Δ_S to $\Delta_{S'}$ is given. Take arbitrary lifts of α_S to $\alpha_{S'}$, γ_S to $\gamma_{S'}$ and of m_S to $m_{S'} \in N_{\Sigma_{S'}}$. (For m_S this is possible by (3.32)ii).) Let $h_{S'} := \alpha_{S'}.m_{S'} + \gamma_{S'}.\Delta_{S'}$
* Consider now $m \in N_\Sigma$, $m = \overline{m_S}$.

  Claim : the homomorphism $h : N_\Sigma \longrightarrow \mathcal{O}_\Sigma \otimes V$; $m \longmapsto h_{S'}$ gives rise to a well - defined element $ob(\xi_S,\Sigma_{S'}) \in N^*/I \otimes V$.

  proof : One has to check several things. For example, when we choose another lift for $\alpha_{S'}$, the difference is of the form $v\otimes\beta$ for some $v \in V$ and β . The quantity $h_{S'}$ then changes by $v\otimes\beta.m$. But by (3.30) this means that the homomorphism h is changed by an element of $I/\int I$, so the class of h in N^*/I stays the same. We omit the further straightforward checks. ⊠

To get an obstruction element only dependent on ξ_S and not on $\Sigma_{S'}$, we have to divide out N^*/I by a subspace that corresponds to $w(T^1_\Sigma)$ in $Tors(\Omega^1_\Sigma)$. To put this subspace in a proper setting we introduce a symmetric bilinear form on N_Σ which is of independent interest.

Definition (3.33) :

Let $\Sigma \hookrightarrow X$ be an admissible diagram, defined by $f \in \int I$. Assume that Σ is reduced and that $T^2_\Sigma = 0$.

The **Hessian H** : $N_\Sigma \times N_\Sigma \longrightarrow \mathcal{O}_\Sigma$ is a symmetric bilinear form defined by the following four steps.

i) Let n and m $\in N_\Sigma$. This means that for all $r \in \mathcal{R}$ we can solve $r.\, n + s(n).\Delta = 0$ and $r.m + s(m).\Delta = 0$ for $s(n)$ and $s(m)$. (Of course, $s(n)$ and $s(m)$ will also depend on $r \in \mathcal{R}$.)

ii) Recall that there is, in general, a pairing $T^1_\Sigma \times T^1_\Sigma \longrightarrow T^2_\Sigma$. The vanishing of this pairing between (the classes of) n and m just means that one can find a p and t such that for all $r \in \mathcal{R}$ one has:

$$r.p + s(n).m + s(m).n + t.\Delta = 0$$

In particular if $T^2_\Sigma = 0$ (as we assumed) this applies.

iii) Because the α - map of f is the zero map, one can solve the equations $\alpha.\, n + \gamma(n).\Delta = 0$ and $\alpha.m + \gamma(m).\Delta = 0$ for $\gamma(n)$ and $\gamma(m)$.

iv) Now put $\mathbf{H}(n,m) := \alpha.p + \gamma(n).m + \gamma(m).n$.

Proposition (3.34) :

The Hessian form **H** has the following properties:

i) $\mathbf{H} : N_\Sigma \times N_\Sigma \longrightarrow \mathcal{O}_\Sigma$ is well - defined, i.e. it does not depend on the choices made in the above steps.

ii) For $\vartheta \in \Theta$ one has $\mathbf{H}(n,\vartheta(\Delta)) = -\vartheta \lrcorner \omega.n$.

iii) For ϑ_1 and $\vartheta_2 \in \Theta$ one has $\mathbf{H}(\vartheta_1(\Delta), \vartheta_2(\Delta)) = -\vartheta_1(\vartheta_2(f))$.

iv) By transposition we get a map $\mathbf{h} : N_\Sigma \longrightarrow N_\Sigma^*$.
The composition $N_\Sigma \xrightarrow{\mathbf{h}} N_\Sigma^* \hookrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$ is equal to the map $-\omega : N_\Sigma \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$ of (3.3).

v) If $f \in I^2$, $f = (h.\Delta).\Delta$ for some matrix h, then $\mathbf{H}(n,m) = -2.h.n.m$.

proof : Statement i) follows by a straightforward check. For example, given f, then the difference $\delta\alpha$ of two choices of α is $\in \mathcal{R}$. This $\delta\alpha$ induces $\delta\gamma$'s such that $\delta\alpha.\, n + \delta\gamma(n).\Delta = 0$ and $\delta\alpha.m + \delta\gamma(m).\Delta = 0$. Then the induced change $\delta\mathbf{H}$ in **H** is given by $\delta\alpha.p + \delta\gamma(n).m + \delta\gamma(m).n$. But by the definition of p this quantity is in the ideal I, hence **H** in $\mathcal{O}_\Sigma$ is independent of the choice of α. Statement ii) can be seen as follows: by differentiating the relations $r.\Delta = 0$ and $r.n + s(n).\Delta = 0$ with respect to $\vartheta \in \Theta$ we get the expressions $r.\vartheta(\Delta) + \vartheta(r).\Delta = 0$ and $r.\vartheta(n) + s(n).\vartheta(\Delta) + \vartheta(r).n + \vartheta(s(n)).\Delta = 0$. Hence $\vartheta(n)$ can be taken as the p of n and $\vartheta(\Delta)$. From $\alpha.\Delta = f$ we get $\gamma(\vartheta(\Delta)) = \vartheta(\alpha) - \vartheta \lrcorner \omega$. Making the substitutions and using $\vartheta(\alpha.n+\gamma(n).\Delta)=0$ we get ii). Statement iii) follows from ii) and expresses the fact that $\mathbf{H}$ is an extension of the *second derivative* of f from vector fields to normal vectors. Statement iv) is just another way to express ii). Statement v) follows by direct calculation. ⊠

Corollary (3.35) :

i) $P_\Sigma(\mathcal{A}) = \ker(\mathbf{h}: N_\Sigma \longrightarrow N^*/I)$; $T^1_\Sigma(X) = \ker(\mathbf{h}: T^1_\Sigma \longrightarrow N^*/I)$.
   where the maps $\mathbf{h}$ are induced by $\mathbf{h} : N_\Sigma \longrightarrow N_\Sigma^*$.

ii) The map $\varphi : P_\Sigma(\mathcal{A}) \longrightarrow I/\int I$ is injective if and only if the map $\mathbf{h} : N_\Sigma \longrightarrow N_\Sigma^*$ injective.

iii) The obstruction $Ob(\xi_S) \in Tors(\Omega^1_\Sigma)/w(T^1_\Sigma) \otimes V$ lifts to an element $ob(\xi_S) \in Coker(\mathbf{h} : T^1_\Sigma \longrightarrow N^*/I) \otimes V \hookrightarrow Tors(\Omega^1_\Sigma)/w(T^1_\Sigma) \otimes V$.

proof : Statement i) follows from (3.34)iv) together with (3.5) and (3.7). Statement ii) follows from the fact that φ is injective if and only if the map $\omega: N_\Sigma \longrightarrow \Omega^1 \otimes \mathcal{O}_\Sigma$ is injective. Now use (3.34)iv) again. Statement iii) is obtained by studying the dependence of $ob(\xi_S, \Sigma_{S'})$ on the chosen lift $\Sigma_{S'}$. We leave the details to the reader. ⊠

The relation between the Hessian form and the number of D_∞ - points appearing in a generic (admissible) perturbation was first noticed by Siersma (see [Si], remark 4.1) in the case that Σ is a smooth curve germ. Pellikaan (see [Pe 2], pp.27-32) generalized this to the case where Σ is a complete intersection curve and more generally to $f \in I^2$ in the case that Σ is *syzygetic* (see [Pe2] for a definition; space curves are syzygetic). He defined the Hessian form in those cases essentially by formula (3.34)v). T. de Jong in [Jo] introduced for a general germ of a hypersurface (X,p) with a one dimensional singular locus Σ and transverse A_1 - singularities an invariant $VD_\infty(X,p)$ (possibly negative! ; for a D_∞ -singularity, $VD_\infty = 1$ and for the triple point $T_{\infty,\infty,\infty}$, $VD_\infty = -2$) and showed an appropriate continuity statement under admissible deformations of $\Sigma \hookrightarrow X$. We will relate this invariant VD_∞ now to the Hessian form $\mathbf{H}$.

Reminder (3.36) :

i) Let $(X,p) \subset (\mathbb{C}^{n+1},0)$ be a germ of hypersurface singularity defined by $f \in \mathcal{O}$ and with one dimensional singular locus Σ and transverse type A_1. Put $\Theta_f := \{\vartheta \in \Theta \mid \vartheta(f)=0\}$ and $\Theta(f) := \Theta_f \otimes \mathcal{O}_\Sigma$. Then the virtual number of D_∞- points, VD_∞ is defined by

$$VD_\infty(X,p) := \dim_{\mathbb{C}}\left(\Theta_{\tilde{\Sigma}} / \Theta(f)\right) - 3.\delta(\Sigma,p).$$

where $\tilde{\Sigma}$ is the normalization of Σ and δ the delta invariant.

ii) Let $Y \subset \mathbb{P}^{n+1}$ be a projective hypersurface of degree d, with a one dimensional singular locus Ξ and transverse type A_1. Then:

$$\sum_{p \in \Sigma} VD_\infty(X,p) = (nd - 2(n+2)).\deg(\Xi) + 4.\chi(\mathcal{O}_\Xi)$$

iii) Let $(X,p) \subset (\mathbb{C}^{n+1}, 0)$ be a germ as under i). Then there exists a $Y \subset \mathbb{P}^{n+1}$ as in ii) with a point $y \in Y$ such that:

a. $(Y,y) \approx (X,p)$

b. $Y - \{y\}$ has only A_∞ and D_∞ - singularities.

proof : For definition i) see [Jo], where one also finds the proof of the continuity under deformation of VD_∞. Result ii) is proved in [J-J]. The proof of iii) involves Sard type of arguments and results on I-finite determinacy (see [Pe3]) and will appear elsewhere. We remark that because the total number of VD_∞ on a projective surface in $\mathbb{P}^3$ is even, one has in general to admit D_∞ - points in the compactification. ⊠

Theorem (3.37) :

Let (X,p) be a germ of a hypersurface with one dimensional singular locus Σ and transverse type A_1. Assume that $T^2_\Sigma = 0$ and that Σ is smoothable and syzygetic. Then:

$$VD_\infty(X,p) = \dim_{\mathbb{C}}\left(N^*_\Sigma / h(N_\Sigma)\right) - \dim_{\mathbb{C}}\left(N^*/I\right) + \dim_{\mathbb{C}}\left(\int I/I^2\right)$$

proof : First we globalize the germ (X,p) to get a $Y \subset \mathbb{P}^{n+1}$ as in (3.36) iii). The compactification of Σ is denoted by Ξ. It is not hard to see the the Hessian form $\mathbf{H} : N_\Sigma \otimes N_\Sigma \longrightarrow \mathcal{O}_\Sigma$ can be globalized to a map $\mathcal{H}_Y : N_\Xi \otimes N_\Xi \longrightarrow \mathcal{O}_\Xi \otimes N_Y$, where N_Y is the normal bundle of Y in $\mathbb{P}^{n+1}$. Now it is checked by calculation that the theorem is true for the A_∞ and the D_∞ - singularity. So by statement ii) of (3.36) it is sufficient to prove that

$$\sum_{p \in \Xi} \dim\left(N^*_{\Xi,p} / h_p(N_{\Xi,p})\right) - \dim\left(N^*_{\Xi,p} / (I/\int I)_p\right) + \dim\left((I/\int I)_p\right) =$$

$$= (nd - 2.(n+2))\deg(\Xi) + 4.\chi(\mathcal{O}_\Xi)$$

where the index p refers to the local invariant of (Y,p) at that point. The global Hessian gives rise to an injective map

$$h_Y : N_\Xi \longrightarrow N^*_\Xi \otimes N_Y$$

with as cokernel a sky scraper sheaf at p of lenght $N^*_{\Xi,p}/h_p(N_{\Xi,p})$. Hence the left hand side of formula above is equal to

$$\chi(N^*_\Xi \otimes N_Y) - \chi(N_\Xi) - \chi(N^*_\Xi) + \chi(I/I^2).$$

Furthermore, $\chi(N^*_\Xi \otimes N_Y) = \chi(N^*_\Xi) + 2.n.\deg(Y).\deg(\Xi)$, by Riemann-Roch, because N_Y is a line bundle. So the statement is equivalent to:

$$\chi(I/I^2) - \chi(N_\Xi) = 4.\chi(\mathcal{O}_\Xi) - 2.(n+2).\deg(\Xi). \qquad (*)$$

But this is a statement that only depends on the curve Ξ. By a theorem of Pellikaan (see [Pe2], thm. 4.5) we know that the statement of the theorem is true if we start with $f \in I^2$ and so the theorem holds for any f. (Another line of argument to see (*) is as follows: (*) is true for smooth Ξ; the conditions syzygetic and $T^2_\Sigma = 0$ give that the left hand side is constant under deformation of Ξ; Σ is assumed to be smoothable, so (*) must be true.) ⊠

Remark (3.38) :

It is desirable to have a more conceptual or *local* proof of the above theorem. We do not know whether the theorem is true for non - smoothable curves Σ. But in any case the conditions on Σ are satisfied when X is a germ of a weakly normal surface in $\mathbb{C}^3$.

The folowing corollary is a partial generalization of [Pe 2], thm.1.13.

Corollary (3.39) :

Under the same assumptions as for (3.37) we have:

$$j(f) = c_{I,e}(f) + VD_\infty(X,p) + \dim_{\mathbb{C}} T^1_\Sigma - \dim_{\mathbb{C}}\left(\textstyle\int I/I^2\right)$$

where $j(f) := \dim_{\mathbb{C}}\left(I/J(f)\right)$ and $c_{I,e}(f) := \dim_{\mathbb{C}}\left(\int I + J(f)/J(f)\right)$.

proof : Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta\otimes\mathcal{O}_\Sigma/\Theta_\Sigma & \longrightarrow & N_\Sigma & \longrightarrow & T^1_\Sigma & \longrightarrow & 0 \\
 & & \downarrow{\lrcorner df} & & \downarrow{h} & & \downarrow{h} & & \\
0 & \longrightarrow & I/\int I & \longrightarrow & N^*_\Sigma & \longrightarrow & N^*/I & \longrightarrow & 0
\end{array}$$

The map $\lrcorner df$ is induced by $\vartheta \in \Theta \longmapsto \vartheta \lrcorner df = \vartheta(f)$

The map $h: N_\Sigma \longrightarrow N^*_\Sigma$ is injective and hence the $\lrcorner df$ is injective. Comparing the indices of the vertical maps of the diagram then gives:

$$\dim_{\mathbb{C}}\left(I/\textstyle\int I + J(f)\right) = \dim_{\mathbb{C}}\left(N^*_\Sigma/h(N_\Sigma)\right) - \dim_{\mathbb{C}}\left(N^*/I\right) + \dim_{\mathbb{C}}\left(T^1_\Sigma\right)$$

The exact sequence

$$0 \longrightarrow (\textstyle\int I + J(f))/J(f) \longrightarrow I/J(f) \longrightarrow I/(\textstyle\int I + J(f)) \longrightarrow 0$$

then gives the corollary when we use (3.37). ⊠

We give another noteworthy formule for VD_∞ for a germ (X,p) of a weakly normal surface in $\mathbb{C}^3$. We let $n : \tilde{X} \longrightarrow X$ be the normalization map, and let $\tilde{\Sigma} = n^{-1}(\Sigma)$ be the inverse image of Σ under the normalization map. We let b be the number of irreducible components of X at p.

Theorem (3.40) :

With the notation as above we have:

$$VD_\infty(X,p) = \mu(\tilde{\Sigma}) - 2.\mu(\Sigma) + 2 - b.$$

proof : We compactify X and Σ as in (3.36) iii) to get a Y and Ξ. Let $\deg(Y) = d$, $\deg(\Xi) = e$. Consider the normalization diagram:

$$\begin{tikzcd} \tilde{\Xi} \arrow[r, hook] \arrow[d] & \tilde{Y} \arrow[d, "n"] \\ \Xi \arrow[r, hook] & Y \end{tikzcd}$$

where $n: \tilde{Y} \longrightarrow Y$ is the normalization map and $\tilde{\Xi} = n^{-1}(\Xi)$.
Note that because X is weakly normal, the ideal sheaf $\mathcal{J}$ of Ξ in $\mathcal{O}_Y$ is equal to the conductor $\mathcal{H}om_Y(n_*\mathcal{O}_{\tilde{Y}}, \mathcal{O}_Y)$.
From the exact sequence $\mathcal{J} \hookrightarrow \mathcal{O}_Y \twoheadrightarrow \mathcal{O}_\Xi$ we get after applying $\mathcal{H}om_Y(-,\mathcal{O}_Y)$ the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow n_*\mathcal{O}_{\tilde{Y}} \longrightarrow \mathcal{E}xt^1_Y(\mathcal{O}_\Xi, \mathcal{O}_Y) \longrightarrow 0 .$$

But $\mathcal{E}xt^1_Y(\mathcal{O}_\Xi, \mathcal{O}_Y) = \mathcal{E}xt^1_Y(\mathcal{O}_\Xi, \omega_Y) \otimes \omega_Y^{-1} = \omega_\Xi \otimes \omega_Y^{-1}$ because ω_Y is a line bundle and using the adjunction formula. Hence:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow n_*\mathcal{O}_{\tilde{Y}} \longrightarrow \omega_\Xi \otimes \omega_Y^{-1} \longrightarrow 0$$

The normalization diagram is a pull - back, hence we also get an exact sequence of the form (see [Str],1.2.3)

$$0 \longrightarrow \mathcal{O}_\Xi \longrightarrow n_*\mathcal{O}_{\tilde{\Xi}} \longrightarrow \omega_\Xi \otimes \omega_Y^{-1} \longrightarrow 0 \qquad (14)$$

So we get $\chi(\mathcal{O}_{\tilde{\Xi}}) = \chi(\mathcal{O}_\Xi) + \chi(\omega_\Xi \otimes \omega_Y^{-1})$

$$\chi(\omega_\Xi \otimes \omega_Y^{-1}) = \chi(\omega_\Xi) + \deg(\mathcal{O}_\Xi \otimes \omega_Y^{-1}).$$

As $\omega_Y \approx \mathcal{O}_Y(d-4)$, we find $\deg(\mathcal{O}_{\Xi} \otimes \omega_Y^{-1}) = e(4-d)$. By Serre duality, $\chi(\mathcal{O}_{\Xi}) = -\chi(\omega_{\Xi})$, hence:

$$\chi(\mathcal{O}_{\tilde{\Xi}}) = e(4-d) \qquad (!)$$

Substitution of this result in formula (3.36) ii) then gives:

$$\sum_{p \in \Xi} VD_{\infty}(Y,p) = 2\left(2.\chi(\mathcal{O}_{\Xi}) - \chi(\mathcal{O}_{\tilde{\Xi}})\right). \qquad (15)$$

One has $2.\chi(\mathcal{O}_{\Xi}) = \mu(\Xi) - \chi_{top}(\Xi)$ (see [Gre], p.149). Remember that $\tilde{\Xi} \longrightarrow \Xi$ is a 2:1 ramified cover, so it is an easy topological matter to relate $\chi(\tilde{\Xi}) - 2.\chi(\Xi)$ to *local* data of the normalization. Using all this we finally find the formula of theorem (3.40). ⊠

In case we have a hypersurface X with singular locus Σ in codimension 1, we will prove that under certain circumstances there is a natural equivalence

$$\mathrm{Def}(\Sigma, X) \xrightarrow{\ \approx\ } \mathrm{Def}(\tilde{X} \longrightarrow X)$$

where $\mathrm{Def}(\tilde{X} \longrightarrow X)$ is the deformation functor of the diagram of the normalization map $\tilde{X} \longrightarrow X$, i.e. the functor of simultaneous normalization of X (see [Bu]). The above equivalence is particularly useful for the study of the deformation theory of normal surface singularities. By projecting such a normal surface singularity into $\mathbb{C}^3$ one gets a hypersurface X together with a curve Σ of double points. By the method of § 3. one can compute the base space of a semi-universal deformation for $\mathrm{Def}(\Sigma, X)$. We will give examples in § 5.

The problem with simultaneous normalization over an infinitesimal basis is that one cannot use the usual construction of integral closure in the total quotient ring to get $\tilde{X}_S$ out of X_S : over $S = \mathrm{spec}(k[\varepsilon]/(\varepsilon^2))$ every element ε/x is integral for $x \in \mathcal{O}_{X_S}$ a non zero divisor. This is reflected in the fact that the natural forgetful transformation $\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(X)$ is not always injective.

It appears that the missing bit of information to construct $\tilde{X}$ out of X is just the *conductor* $C := \mathcal{H}om_X(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. We can consider $\mathcal{O}_{\tilde{X}}$ as a module over $\mathcal{O}_X$. When we deform $\mathcal{O}_X$ flat over S to an $\mathcal{O}_{X_S}$, it turns out that deforming the $\mathcal{O}_X$-module $\mathcal{O}_{\tilde{X}}$ to an S-flat $\mathcal{O}_{X_S}$ -module $\mathcal{O}_{\tilde{X}_S}$ is equivalent to deforming the conductor C flat to an C_S. However, the conductor C is a very special ideal in $\mathcal{O}_X$: the fact that $\mathcal{O}_{\tilde{X}}$ carries a *ring structure* is equivalent to:

Ring Condition (R.C.)

$$\mathcal{H}om_X(C, C) \xhookrightarrow{\ \approx\ } \mathcal{H}om_X(C, \mathcal{O}_X)$$

The last statement makes sense over any basis S, and it turns out that elements of $\mathrm{Def}(\tilde{X} \longrightarrow X)(S)$ correspond to deformations of X and C to X_S and C_S for which C_S still satisfies the corresponding condition (R.C.). To be precise, one has the following theorem:

Theorem (4.1) :

Let $\tilde{X} \longrightarrow X$ be a finite surjective and generically injective mapping. Let Σ be the subspace of X defined by the conductor ideal $C = \mathcal{H}om_X(\mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. Assume that:

i) $\tilde{X}$ is Cohen - Macaulay

ii) X is Gorenstein

Then there is a natural equivalence of functors

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(\Sigma \hookrightarrow X, \mathrm{R.C.})$$

Here the second functor is deformations of the diagram $\Sigma \hookrightarrow X$ for which the ideal of Σ_S in X_S satisfies condition R.C.

The next thing to do is to relate (R.C.) to admissibility. For this we need some more conditions on X and Σ.

Theorem (4.2) :

Let $\Sigma \hookrightarrow X$ be an admissible diagram. Assume that;

i) X is a hypersurface

ii) Σ is Cohen-Macaulay of codimension 2.

iii) Σ is reduced.

Let $\Sigma_S \hookrightarrow X_S$ be any deformation of this diagram over S. Then equivalent are:

i) the map $\alpha_S : N_{\Sigma_S} \longrightarrow \mathcal{O}_{\Sigma_S}$ is the zero map (see (3.31)).

ii) the ideal I_S of Σ_S satisfies (R.C.).

iii) the diagram $\Sigma_S \hookrightarrow X_S$ is admissible.

When we combine theorems (4.1) and (4.2) we get the following:

Theorem (4.3) :

Let $\tilde{X} \longrightarrow X$ be a finite, generically injective map. Let Σ be the subspace of X defined by the conductor. Assume that:

i) $\tilde{X}$ is Cohen-Macaulay.

ii) X is a hypersurface.

iii) Σ is reduced.

Then there is a natural equivalence of functors

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(\Sigma, X).$$

To complete the picture we state one other theorem

Theorem (4.4) :

Under the same conditions as in theorem (4.3) one has that the natural forgetful transformation

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(\tilde{X})$$

is smooth.

Theorems (4.3) and (4.4) together imply that the base space of the semi-universal deformation of $\tilde{X}$ is, up to a smooth factor, the same as the base space of the functor Def(Σ,X). So the whole complexity of deformations of normal surfaces is reflected in the theory of admissible deformations of weakly normal (i.e. generically transverse A_1) surfaces in $\mathbb{C}^3$.

The rest of this paragraph is devoted to the proofs of the above stated theorems. For notational convenience and clarity of exposition we change from geometric language to algebraic language.

Let R and S rings as in § 0. Let $QR \supset R$ be the total quotient ring of R.

Definition (4.5):

A *fractional ideal* is a finitely generated R - module M such that:

i) $M \subset QR$

ii) M contains a non - zero divisor.

Lemma (4.6) :

i) If $\overline{M}$ is a fractional ideal in $Q\overline{R}$ and M is an S - flat R-module, then M is a fractional ideal in QR.

ii) Let M and N be fractional ideals in QR. Then $Hom_R(M,N)$ is also a fractional ideal and can be identified with $\{x \in QR \mid x.M \subset N\}$.

proof : Left as an exercise to the reader. We only note that the map from $Hom_R(M,N)$ to QR is given by: $(\varphi : M \longrightarrow N) \longmapsto \varphi(m)/m$ (m non - zero divisor in M). ⊠

Proposition (4.7):

Let R be a Gorenstein ring over S, i.e. $\omega_{R/S} \approx R$. Then the duality functor $M \longmapsto M^{\vee} := Hom_R(M,R)$ on the category of R-modules has the following properties:

i) it converts fractional ideals into fractional ideals.

ii) it converts MCM's over S to MCM's over S (see (0.4)).

iii) it is an inclusion reversing involution on the category of fractional MCM's over S

iv) it commutes with specialization for MCM's, i.e. $(\overline{M})^{\vee} = \overline{(M^{\vee})}$

proof : i) follows from (4.6)ii) and ii) follows from the Gorenstein assumption and proposition (0.10)i). The involutivity iii) results from (0.10)ii), whereas iv) follows from (0.7)iii) (and (0.9)) ⊠

When a fractional MCM happens to be an overring $\tilde{R}$ of the ring R, then its dual module $C = \mathrm{Hom}_R(\tilde{R},R)$ is an ideal in R, called the *conductor* of $\tilde{R}$ over R. This conductor has a special property:

Proposition (4.8) :

Let $\tilde{R} \supset R$ be a fractional MCM over S and let $C \subset R$ be its dual module. Then equivalent are:

i) $\tilde{R}$ is a ring (with ring structure induced from $\tilde{R} \subset QR$)

ii) The ideal C satisfies the *Ring Condition (R.C.)* , i.e. the natural inclusion map

$$\mathrm{Hom}_R(C,C) \hookrightarrow \mathrm{Hom}_R(C,R)$$

is an isomorphism.

proof : ii) $\Rightarrow$ i) : as we have $\tilde{R}=\mathrm{Hom}_R(C,R)$ by (4.7)iii) we see that if $\mathrm{Hom}_R(C,C) \approx \mathrm{Hom}_R(C,R)$ then $\tilde{R}$ gets the ring structure as the endomorphisms of the R - module C.

i) $\Rightarrow$ ii) : for this we need the 'duality lemma for finite maps' (see [Ha], ex. 6.10, p.239) or 'change of rings isomorphism'

$$\mathrm{Hom}_{\tilde{R}}(M,\mathrm{Hom}_R(\tilde{R},N)) \approx \mathrm{Hom}_R(M,N) .$$

(Here M is any finitely generated $\tilde{R}$ - module and N any R - module.) Now it easy to see that the conductor C is also an $\tilde{R}$ - ideal, so we can take $M=C$ and $N=R$ in the above formula to get $\mathrm{Hom}_{\tilde{R}}(C,C)= \mathrm{Hom}_R(C,R)$. But clearly one has $\mathrm{Hom}_R(C,C) \supseteq \mathrm{Hom}_{\tilde{R}}(C,C)$. Combining these last two facts we get $\mathrm{Hom}_R(C,C)=\mathrm{Hom}_R(C,R)$. ⊠

proof of theorem (4.1) : Start with a map $\tilde{X} \longrightarrow X$ as in the statement of the theorem. Consider a deformation X_S over S. Then the category of diagrams $\tilde{X}_S \longrightarrow X_S$ corresponds exactly to the fractional MCM's for the ring $R = \mathcal{O}_{X_S}$ having $\mathcal{O}_{\tilde{X}}$ as special fibre. By (4.7) and (4.8) the duality functor transforms these into diagrams $\Sigma_S \hookrightarrow X_S$ for which the ideal satisfies (R.C.). ⊠

We now turn to the proof of theorem 4.2 . Let P be the local ring of the ambient space, which is regular over the local ring S of the base. We assume that X_S is a hypersurface, so the local ring R of X_S is of the form $R = P/(F)$, where $F \in P$ is a non - zero divisor. Let I be the ideal of Σ_S in the ring R, so the local ring of Σ_S is R/I. As a subspace of the ambient space, Σ_S is given by an ideal I_P in the ring P. By assumption, Σ_S is CM over S of codimension 2. This implies that the equations of Σ_S are of a special form.

Lemma (4.9) :

There exists a free resolution of I_P as a P - module of the following form:

$$0 \longrightarrow P^r \xrightarrow{\ M\ } P^{r+1} \xrightarrow{\ \Delta\ } I_P \longrightarrow 0$$

Here M is a certain $r \times (r+1)$ matrix and the generators Δ_i of I_P (i.e. the components of the map Δ) are given by the the $r \times r$ minors of M.

proof : The resolution of $\bar{I}_P$ over $\bar{P}$ has the form as above, by the theorem of Hilbert-Burch-Schaps (see [Ar 1], pp.16-17.) As I_P is S - flat by assumption, we find a resolution as above over the ring P. ⊠

Because Σ_S is a subspace of X_S we have $F \in I_P$, i.e. we can write :

$$F = \sum_{i=0}^{r} \alpha_i \cdot \Delta_i$$

Proposition (4.10) :

There is a free resolution of I over P of the form:

$$0 \longrightarrow P^{r+1} \xrightarrow{\ \tilde{M}\ } P^{r+1} \xrightarrow{\ \Delta\ } I \longrightarrow 0$$

Here the matrix $\tilde{M}$ is obtained from the matrix M by adjoining the vector $(\alpha_0 , \alpha_1 , \ldots , \alpha_r)$ as zeroth column, so $\det(\tilde{M}) = F$.

proof : As we have $R/I = P/I_P$ we get an exact sequence of the form:

$$0 \longrightarrow P.F \longrightarrow I_P \longrightarrow I \longrightarrow 0$$

The result now follows from (4.9) and the following commutative diagram from which one can conclude the exactness of the bottom row.

$$\begin{tikzcd}
0 \arrow[r] & P^r \arrow[r, "M"] \arrow[d, hook] & P^{r+1} \arrow[r, "\Delta"] \arrow[d, "="] & I_P \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & P^{r+1} \arrow[r, "\tilde{M}"] & P^{r+1} \arrow[r, "\Delta"] & I \arrow[r] & 0
\end{tikzcd}$$

⊠

Corollary (4.11) :

i) The module I has a 2 - periodic resolution over the ring R of the following form:

$$\ldots \xrightarrow{\Phi} \mathcal{G} \xrightarrow{\Psi} \mathcal{F} \xrightarrow{\Phi} \mathcal{G} \longrightarrow I \longrightarrow 0 .$$

Here $\Phi = \tilde{M}$ mod F and $\Psi = \bigwedge^r \Phi$ is the *Cramer matrix* of Φ, i.e. the matrix having as entries the $r \times r$ - minors of Φ. $\mathcal{F}$ and $\mathcal{G}$ are free R - modules of rank (r+1).

ii) The dual module $I^{\vee} = \mathrm{Hom}_R(I, R)$ has a 2 - periodic resolution over the ring R of the form:

$$\ldots \xrightarrow{\Phi^{\vee}} \mathcal{F}^{\vee} \xrightarrow{\Psi^{\vee}} \mathcal{G}^{\vee} \xrightarrow{\Phi^{\vee}} \mathcal{F}^{\vee} \longrightarrow I^{\vee} \longrightarrow 0$$

Here $\Phi^{\vee}$ is the transpose of the map Φ.

iii) One has $I \approx \mathrm{Coker}(\Phi) \approx \mathrm{Ker}(\Phi) \approx \mathrm{Im}(\Psi)$
and $I^{\vee} \approx \mathrm{Coker}(\Phi^{\vee}) \approx \mathrm{Ker}(\Phi^{\vee}) \approx \mathrm{Im}(\Psi^{\vee})$.

proof : It is a standard matter to come from the the resolution over the ring P to a resolution over R. (Matrix factorization, see [Ei].) Hence we get i). ii) is obtained by dualizing i) and using iii), which follows from the 2 - periodicity of the complex under i). ⊠

Let $N := \mathrm{Hom}_P(I_P, P/I_P)$ be the normal bundle of Σ_S in the ambient space. In §3. we already encountered the so-called α-map

$$\begin{array}{rccc} \alpha : & N & \longrightarrow & R/I \\ & (\varphi: \Delta_i \longmapsto n_i) & \longmapsto & \sum \alpha_i . n_i \end{array}$$

The pivotal result about the α - map is the following.

Theorem (4.12) :

With the notation as above, the following are equivalent:

i) $\mathrm{Hom}_R(I,I) = \mathrm{Hom}_R(I,R)$, i.e. I satisfies (R.C.).

ii) the entries of the matrix Ψ are in I.

iii) the α - map $\alpha : N \longrightarrow R/I$ is the zero map.

proof : By (4.11) iii) , an element $\delta \in \mathrm{Hom}_R(I,R) = I^{\vee}$ is represented by an element δ' of $\mathcal{G}^{\vee}$ in $\mathrm{Im}(\Psi^{\vee})$. To evalute δ on an element $i \in I$, represent i by an element $i' \in \mathcal{G}$ and let δ' act on i' . As $\delta' \in \mathrm{Im}(\Psi^{\vee})$ we see that the ideal generated by the matrix elements of $\Psi^{\vee}$ (or Ψ) is the ideal generated by the $\delta(i)$, $\delta \in I^{\vee}$, $i \in I$. Hence i) $\Leftrightarrow$ ii).

Because Σ_S is CM over S of codimension 2, a generating set of N can be obtained by 'perturbing' the matrix M (see [Ar1], p.16-21). To be more precise, let λ be any $r\times(r+1)$ matix with entries in R. Then one has:

$$\textstyle\bigwedge^r(M+\varepsilon.\lambda) = \bigwedge^r(M) + \varepsilon.\bigwedge^{r-1}(M)\wedge\lambda \mod \varepsilon^2 .$$

So λ gives rize to a normal vector $n^{\lambda} \in N$ corresponding to the homomorphism n^λ : $I_P \longrightarrow R/I$

$$\Delta_i \longmapsto (\bigwedge^{r-1}(M) \wedge \lambda)_i \ .$$

A little calculation then shows that

$$\alpha(n^\lambda) = \mathrm{Trace}\ (\tilde{\Psi}\,.\,\lambda)$$

where $\tilde{\Psi}$ is the matrix obtained from Ψ by erasing the 0th row. When we let λ run over the elementary matrices e_{ij} , $1 \le i \le r$, $0 \le j \le r$ we get

$$\alpha(n^{e_{ij}}) = \Psi_{ij}$$

and hence the equivalence between ii) and iii). ⊠

Remark (4.13):

Property ii) in (4.12) can be reformulated as a property of the matrix $\tilde{M}$ or Φ and is called the *Rank Condition* in [Ca] and [M-P]: an $(r+1)\times(r+1)$ matrix Φ is said to satisfy the Rank Condition if the ideal generated by the $r\times r$ - minors of Φ is the same as the ideal generated by the $r\times r$ - minors of the matrix obtained from Φ by deleting the first (zeroth) column. Catanese [Ca] also calls this the Rouché - Capelli property. In any case, the abbreviation (R.C) seems extremely appropriate. For a discussion of the equations defining the ring $\mathrm{Hom}_R(I, I)$ we refer to [Ca] and [M-P].

proof of theorem (4.2): By (4.12) we have that (R.C) is equivalent to the condition that the α - map is the zero map. By theorem (3. 31) we have: If $\bar{I}$ is a *reduced* ideal in $\bar{R}$ then

$$\alpha \text{ - map is the zero map} \Leftrightarrow (F, J_F) \subset I$$

Hence, under the assumptions of (4.2) we have indeed :

$$I = I_S \text{ satisfies (R.C)} \Leftrightarrow \alpha = \alpha_S \text{ - map is zero map} \Leftrightarrow$$

$$\Leftrightarrow \Sigma_S \hookrightarrow X_S \text{ is admissible.}$$ ⊠

Remark (4.14) :

Theorem (4.3) states that in case of a finite generically injective map $\tilde{X} \longrightarrow X$ between a Cohen - Macaulay space $\tilde{X}$ and a hypersurface X with a *reduced* conductor we have an equivalence of functors

$$\mathrm{Def}(\tilde{X} \longrightarrow X) \xrightarrow{\approx} \mathrm{Def}(\Sigma, X)$$

where Σ is given the conductor structure.

We have seen in (2.6) an example where $\mathrm{Def}(\Sigma, X) \longrightarrow \mathrm{Def}(X)$ is not injective, although Σ was reduced and X had an isolated singular point. One calculates that in (2.6) the conductor of the normalization map is just the maximal ideal. Hence, by theorem (4.3) this means that (2.6) also affords an example where $\mathrm{Def}(\tilde{X} \longrightarrow X) \longrightarrow \mathrm{Def}(X)$ is not injective, i.e. an example of non- uniqueness of the normalization mapping in deformational context. This can also been seen directly: Consider the normalization mapping:

$$\begin{array}{ccc} X & \longleftarrow & \tilde{X} \\ k[x,y]/(x^3+y^2) & \hookrightarrow & k[t] \\ x & \longmapsto & -t^2 \\ y & \longmapsto & t^3 \end{array}$$

This mapping can be deformed non - trivially over $k[\varepsilon]/(\varepsilon^2)$ without changing the image by $x \longmapsto -t^2 - 2\varepsilon$; $y \longmapsto t^3 + 3\varepsilon t$. This corresponds exactly to the deformation in (2.6). More generally, for the normalization mapping $\tilde{X} \longrightarrow X$ of a curve germ X one has the following result (see [Bu], p. 82):

$$\dim\Big(\ker\Big(\mathrm{Def}(\tilde{X} \longrightarrow X)(k[\varepsilon]/(\varepsilon^2)) \longrightarrow \mathrm{Def}(X)(k[\varepsilon]/(\varepsilon^2))\Big)\Big) = m - r$$

where m is the multiplicity and r the number of branches of X at the special point of X.

Remark (4.15) :

Let I be an MCM ideal in a hypersurface ring R satisfying (R.C) and let $\tilde{R} = \mathrm{Hom}_R(I,I) = I^\vee \supset R$ the ring extension of R belonging to it. As the complex (4.11) i) is 2 - periodic, it is not hard to compute all the higher Ext' s of I. The result is:

$$\left.\begin{array}{ll} * & \mathrm{Hom}_R(I,I) = \tilde{R} \\ * & \mathrm{Ext}_R^{2k+1}(I,I) = N \\ * & \mathrm{Ext}_R^{2k}(I,I) = \tilde{R}/I \end{array}\right\} \quad k = 0, 1, 2, \ldots$$

In fact, taking $\mathrm{Hom}_R(I,-)$ to the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

we get a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(I,I) \longrightarrow \mathrm{Hom}_R(I,R) \longrightarrow \mathrm{Hom}_R(I,R/I) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}_R^1(I,I) \longrightarrow \mathrm{Ext}_R^1(I,R) \longrightarrow \ldots$$

As I is assumed to be MCM (over S) and $R \approx \omega_{R/S}$ we have that $\mathrm{Ext}_R^1(I,R) = 0$. Hence, I satisfies (R.C) $\Leftrightarrow$ $N \approx \mathrm{Ext}_R^1(I,I)$, where $N = \mathrm{Hom}_R(I,R/I)$ is the normal bundle of Σ_S in X_S . Note that this normal bundle is also equal to the normal bundle of Σ_S in the ambient space $\mathrm{Hom}_P(I_P, R/I)$ if the α - map is zero.

The 2 - periodicity gives that $\mathrm{Ext}_R^2(I,I)$ is a quotient of $\mathrm{Hom}_R(I,I)$. One can check that annihilator is precisely I.

Thus we get a Yoneda Ext - pairing

$$\begin{array}{rcl} \mathrm{Ext}_R^1(I,I) \times \mathrm{Ext}_R^1(I,I) & \longrightarrow & \mathrm{Ext}_R^2(I,I) \\ Y \; : \; N \times N & \longrightarrow & \tilde{R}/I \end{array}$$

In general, this pairing is not symmetric. One can proof that the symmetrization Y^+ of Y takes values in R/I and can be identified with the Hessian **H** of § 3 (see (3.32)). We do not have an interpretation of the anti-symmetric part Y^- of Y, although we expect it to contain interesting new information.

In the deformation theory of X together with the module I one encounters natural maps $T^i_X \longrightarrow Ext^{i+1}_X(I,I)$. We only state:

* $T^0_X \longrightarrow Ext^1_X(I,I)$ is the zero - map.
* $T^1_X \longrightarrow Ext^2_X(I,I)$ has as kernel $I/(f,J(f))$

Indeed, for $g \in I$ one can lift the module I over the hypersurface with equation $f + \varepsilon.g$, $\varepsilon^2 = 0$. But it requires extra conditions on g that the deformed I satisfies (R.C) or stays admissible (see § 3).

To conclude this section we give a result that implies theorem (4.4).

Proposition (4.16) :

Let $\tilde{X} \longrightarrow X$ be a mapping and $X \subset Y$ an embedding of X in a space Y smooth over the base field. Then :

i) There is a natural transformation of functors

$Def(\tilde{X} \longrightarrow X) \longrightarrow Def(\tilde{X} \longrightarrow Y)$

ii) The natural transformation $Def(\tilde{X} \longrightarrow Y) \longrightarrow Def(\tilde{X})$ is smooth.

iii) If $\tilde{X}$ is Cohen-Macaulay, X is a hypersurface in Y and the map $\tilde{X} \longrightarrow X$ is generically injective, then the transformation in i) is an equivalence of functors.

Sketch of proof : Let $\left(\tilde{X}_S \longrightarrow X_S\right) \in Def(\tilde{X} \longrightarrow X)$. One can extend the inclusion $X \subset Y$ to an inclusion $X_S \subset Y_S$ $(= Y \times Spec(S))$, because all deformations can be realized by embedded deformations. Now the composition $\tilde{X}_S \longrightarrow X_S \subset Y_S$ determines a well- defined element of $Def(\tilde{X} \longrightarrow Y)$. This gives i). Statement ii) follows immediately from the smoothness of Y. For statement iii) we to construct an inverse to transformation i), i.e. an *image functor*. Let $\left(\tilde{X}_S \longrightarrow Y_S\right)$ be an element of $Def(\tilde{X} \longrightarrow Y)$. Let $\tilde{R}$ be the local ring of $\tilde{X}_S$ and P the local ring of Y. Because $\tilde{R}$ is Cohen-Macaulay (over S), it has a presentation as a P - module as the cokernel of a square matrix $\tilde{N}$ (in fact, it is the transpose of the matrix $\tilde{M}$ of (4.10)). Now define X_S to be the hypersurface in Y_S given by the equation $\det(\tilde{N}) = 0$. It is now easy to check that $\left(\tilde{X}_S \longrightarrow X_S\right) \in Def(\tilde{X} \longrightarrow X)(S)$. ⊠

Consider a germ $X \subset \mathbb{C}^3$ of a weakly normal surface and let Σ be the reduced singular locus of X. By theorem (4.3) $Def(\Sigma,X)$ is naturally equivalent to $Def(\tilde{X} \longrightarrow X)$, where $n: \tilde{X} \longrightarrow X$ is the normalization map. Furthermore, by theorem (4.4) the natural forgetful transformation $Def(\tilde{X} \longrightarrow X) \longrightarrow Def(\tilde{X})$ is smooth. Consequentely, the space $T^1_{\tilde{X}}$ of first order deformations of $\tilde{X}$ is a quotient of the space $T^1(\Sigma,X)$ of first order admissible deformations (see § 3). So in order to describe $T^1_{\tilde{X}}$ in terms of $T^1(\Sigma,X)$ we have to identify those first order admissible deformations which deform $\tilde{X}$ trivially. Recall that by theorem (3.15) one has $T^1(\Sigma,X) = P_X(\mathcal{A})/(f,J(f))$, where $f \in \mathbb{C}\{x,y,z\}$ is an equation for X.

Theorem (5.1) :

In the situation as above one has:

$$T^1_{\tilde{X}} = T^1(\Sigma,X)/\mathcal{O}_{\tilde{X}}.J(f) \ \left(= P_X(\mathcal{A})/(f, \mathcal{O}_{\tilde{X}}.J(f))\right)$$

Here $\mathcal{O}_{\tilde{X}}.J(f)$ is the ideal in $\mathcal{O}_{\tilde{X}}$ generated by $J(f)$.

proof : Let (Φ,Ψ) the matrix factorization as in (4.11). So we have $\mathcal{O}_{\tilde{X}} = Coker(\Phi^\vee)$. If we choose a basis $1 = u_0, u_1, \ldots, u_t$ for $\mathcal{F}^\vee$ we get an embedding $i : \tilde{X} \hookrightarrow Spec(\mathbb{C}\{x,y,z\} \otimes \mathbb{C}[u_1, u_2, \ldots, u_t]) =: Y$. Part of the equations of $\tilde{X} \subset Y$ is given by:

$$\sum_{i=0}^{r} u_i . \Phi^\vee_{ij} = 0 \qquad j = 0,1,2,\ldots,t. \qquad (*)$$

(For a more complete discussion of the equations of $\tilde{X}$ in Y we refer to Catanese [Ca] and Mond & Pellikaan [M-P].) To get $T^1_{\tilde{X}}$ out of $T^1(\Sigma,X)$ we have divide out the action of all the vector fields on Y, i.e. $T^1_{\tilde{X}} = P_X(\mathcal{A})/(f, \Theta_Y(f))$. As an $\mathbb{C}\{x,y,z\}$ - module, Θ_Y is generated by $u_k . \partial/\partial u_l$, $u_k .\partial/\partial x$, $u_k .\partial/\partial y$, $u_k .\partial/\partial z$. Consider the matrix $\Phi^{\vee(kl)}$ ($k = 0, \ldots, t$; $l = 1, \ldots, t$) with entries:

$$\Phi^{\vee(kl)}_{ij} = \Phi^{\vee}_{ij} + \varepsilon . \delta_{ik} . \Phi^{\vee}_{lj}$$

where δ_{ij} is the Kronecker delta. This matrix satisfies (R.C) over the ring $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, as is easily checked. As $\det(\Phi^{\vee(kl)}) = f + \varepsilon.\delta_{kl}.f$, this gives a trivial deformation of X. But by differentiating (*) with respect to $u_k . \partial/\partial u_l$ we see that the effect of this vector field on the embedding $\tilde{X} \subset Y$ is just described by the matrix $\Phi^{\vee(kl)}$. Hence, to get $T^1_{\tilde{X}}$ from $T^1(\Sigma,X)$ we only have to divide out $\mathcal{O}_{\tilde{X}}.J(f)$. ⊠

In general it is not so easy to use this direct description of $\mathcal{O}_{\tilde{X}}.J(f)$. In fact we have another description of $\mathcal{O}_{\tilde{X}}.J(f) \hookrightarrow \mathcal{O}_{\tilde{X}}$. We can expand the 1 - forms ω_j of (3.1)(5) into a matrix $\omega = (\omega_{jl})$ defined by

$$\partial f/\partial x_j = \sum_{l=0}^{t} \omega_{jl} . \Delta_l \qquad (j = 0, 1, 2, \ldots, n)$$

Theorem (5.2) :

With the notation and the assumptions as above one has that $\mathcal{O}_{\tilde{X}}.J(f)$ is the ideal generated by the entries of the matrix $\omega . \Psi^{\vee}$.

proof : The elements u_m ($m=0, 1, 2, \ldots, t$) of $\mathcal{O}_{\tilde{X}}$ correspond to the homomorphisms $[u_m]: \Delta_l \longmapsto \Psi^{\vee}_{lm}$ of $\mathrm{Hom}_X(I,I) = \mathcal{O}_{\tilde{X}}$.
So $[u_m \partial f/\partial x_k]: \Delta_l \longmapsto \sum \omega_{kl}.\Delta_l.\Psi^{\vee}_{lm}$. As we have relations of the form $\Psi^{\vee}_{lm}.\Delta_l = \Psi^{\vee}_{lm}.\Delta_l$ (modulo f) we see that the homomorphism $[u_m.\partial f/\partial x_k]$ corresponds to multiplication by $\sum \omega_{kl}.\Psi^{\vee}_{lm} \in \mathcal{O}_{\tilde{X}}$. ⊠

To compute $T^1_{\tilde{X}}$ we can use any X which has $\tilde{X}$ as normalization. We will give some examples.

Examples (5.3):

1) $f = z^2 - y^2(y + x^k)$. This is the $J_{k,\infty}$ - singularity (see [Si]). The normalization $\tilde{X}$ is smooth, and the ideal $P_X(\mathcal{A}) = (x^k y, z, y^2)$. The matrix factorization of f is given by:

$$\Phi^\vee = \begin{pmatrix} z & y(y+x^k) \\ y & z \end{pmatrix} ; \quad \Psi^\vee = \begin{pmatrix} z & -y(y+x^k) \\ -y & z \end{pmatrix}$$

Furthermore, one can take for the ω - matrix the following:

$$\omega = \begin{pmatrix} 0 & k.x^{k-1}y \\ 0 & y(3y+2x^k) \\ 2 & 0 \end{pmatrix}, \text{ so } \omega.\Psi^\vee = \begin{pmatrix} -k.x^{k-1}y^2 & k.x^{k-1}yz \\ -y(3y+2x^k) & z(3y+2x^k) \\ 2.z & -2.y(y+x^k) \end{pmatrix}$$

So indeed $\mathcal{O}_{\tilde{X}}.J(f) = (y^2, x^k y, z)$ and thus $T^1_{\tilde{X}} = 0$.

2) $f = xz^2 - y^2(y+x^k)$. This is the $Q_{k,\infty}$ - singularity (see [Si]). We take

$$\Phi^\vee = \begin{pmatrix} xz & y(y+x^k) \\ y & z \end{pmatrix}$$

Because $\mathcal{O}_{\tilde{X}} \approx \mathrm{Coker}(\Phi^\vee)$, we see that $\mathcal{O}_{\tilde{X}}$ is generated as $\mathcal{O}_X$ - module by 1 and $u := xz/y$. The equations of $\tilde{X}$ in $\mathbb{C}^4$ are :

$$u^2 = x.(y+x^k) \quad ; \quad uy = xz \quad ; \quad uz = y(y+x^k) \quad .$$

The inverse image of the singular locus under the normalization map is given by $u^2 = x^{k+1}$; $y = 0$; $z = 0$, and so is an A_k - singularity. The coordinate transformation $u' = u$; $x' = x$; $y' = y + x^k$; $z' = z + ux^{k-1}$ transforms the equations into $(u')^2 = x'.y'$; $u'.y' = x'.z'$; $u'.z' = (y')^2$. Hence, $\tilde{X}$ is isomorphic to the cone over the rational normal curve of degree 3. It is well known that $\dim T^1_{\tilde{X}} = 2$ (see [Pi]). One has $(\mathcal{O}_{\tilde{X}}.J(f), f) = (xy^2, xz, z^2, 3y^2 + 2x^k y)$ and $P_X(\mathcal{A}) = (y^2, yz, xz, x^k y)$. (Computations left to the reader). Hence $T^1_{\tilde{X}}$ is represented by the classes of y^2 and yz. We leave it to the reader to do more examples, e.g. one could try $f = xyz + y^{p+3} + z^{q+3}$.

3) $f = (yz)^2 + (zx)^2 + (xy)^2$. This is a continuation of example (3.16)4) and (3.24). Here one has $\tilde{X} = \mathrm{Cone}(\,|\mathcal{O}(4)| : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4\,)$. By Pinkham, (see [Pi]), $\dim T^1_{\tilde{X}} = 4$. We already know from (3.16) that:

$$P_X(\mathcal{A}) = (y^2z, yz^2, z^2x, zx^2, x^2y, xy^2, xyz) = m^3 \cap I.$$

One calculates that $(\mathcal{O}_{\tilde{X}}.J(f), f) = m^4 \cap I + J(f)$. A basis for $T^1_{\tilde{X}}$ is represented by $\{xyz, y^2z + yz^2, z^2x + zx^2, x^2y + xy^2\}$.

Let $\tilde{X}$ be a germ of a normal surface singularity. Consider a *smoothing* of $\tilde{X}$, i.e. a deformation $\tilde{X}_S \longrightarrow S$ such that for general $s \in S$ the fibre $\tilde{X}_s$ is smooth. Let S' be the component of the semi--universal deformation on which this smoothing occurs. A theorem of J. Wahl (proved under a condition of globalizability, which Looijenga (see [Lo]) proved to be always fulfilled) relates the dimension of S' to the topology of $\tilde{X}_s$. To formulate this result, let $\pi: Y \longrightarrow \tilde{X}$ be the minimal resolution of $\tilde{X}$, let E be the exceptional divisor and $p_g = \dim(R^1\pi_* \mathcal{O}_Y)$ the geometric genus.

Theorem (J. Wahl, [Wa2], 3.13c)

$$\dim(S') = \left(\dim H^1(\Theta_Y) - 14 p_g - 2.\chi(E)\right) + 2.\chi(\tilde{X}_s)$$

where χ is the topological Euler characteristic. ⊠

Note that the term in the big brackets only depends on $\tilde{X}$.

Now let $X \subset \mathbb{C}^3$ a germ of a weakly normal surface and let Σ be its reduced singular locus. Corresponding to the notion of a smoothing of the normalization $\tilde{X}$ there is the notion of a *disentanglement* of (Σ, X).

Definition (5.6):

A *disentanglement* of $\Sigma \hookrightarrow X$ is an admissible deformation $\Sigma_S \hookrightarrow X_S$ over a basis S such that for a general $s \in S$ the fibre X_s has only A_∞, D_∞ and $T_{\infty,\infty,\infty}$ - singularities (ordinary double curve, ordinary pinch point, ordinary triple point, c.f. 3.17).

Clearly, the normalization $\tilde{X}_S$ of a disentanglement of X is a smoothing of $\tilde{X}$. We want to compute $\chi(\tilde{X}_s)$ in terms of invariants of X. We choose a Milnor representative for X_S (see [Si]). Let F be the Milnor fibre of X. We can compare F with X and with X_s.

Lemma (5.7) :

With the notations as above we have for a disentanglement:

i) $\chi(\tilde{X}_s) = \chi(X_s) + \chi(\Sigma_s) - D + T$

ii) $\chi(F) = \chi(X_s) - \chi(\Sigma_s) + 2D$

iii) $\chi(\Sigma_s) = 2T - \mu(\Sigma) + 1$

where D is the number of D_∞ and T the number of $T_{\infty,\infty,\infty}$ - points in the disentanglement.

proof : This is a simple excercise in topology. For i) one has to realize that $\tilde{X}_s \longrightarrow X_s$ is 1 - 1 except over Σ where it is 2 - 1 except over the pinch points and the triple points. For ii) one compares F with X_s. As one knows the local structure of the degeneration this is easy. For iii) one has to use the fact that Σ_S is a flat family over S and in such a family the jump in topology is the jump in Milnor numbers (see [B-G]). ⊠

Corollary (5.8) :

i) $$\chi(\tilde{X}_s) = \Big(j(f) - 2.VD_\infty(X) - \mu(\Sigma) + 2 \Big) - T$$

ii) dim(S') + 2T is an invariant of X and does not depend on the particular disentanglement chosen.

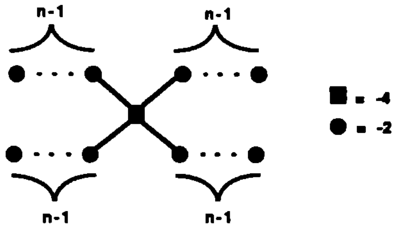
proof : By [Jo], thm.3.2 one has $\chi(F) = j(f) + VD_\infty + \mu(\Sigma)$. Furthermore, one has $VD_\infty = D - 2T$. Combining this with lemma (5.7) we get i). Statement ii) now follows from i) together with the quoted formula of J.Wahl. ⊠

Example (5.9) :

1) X of type D_∞ : $\tilde{X}$ is smooth, hence $\chi(\tilde{X}_s) = 1$. One has $j(f) = 1$, $VD_\infty = 1$ and $\mu(\Sigma) = 0$.

2) $f = (yz)^2 + (zx)^2 + (xy)^2$, $X = \{f = 0\}$. Here the normalization $\tilde{X}$ is Cone($|\mathcal{O}(4)| : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4$). The base space of $\tilde{X}$ has two smoothing components, one of dimension 1 and one of dimension 3 (see [Pi]). One calculates $j(f) = 10$, $VD_\infty = 4$, $\mu(\Sigma) = 2$. For the one dimensional smoothing component we have T = 1 , for the three dimensional component one has T = 0.

The rest of this paragraph is devoted to the study of a particular class of normal surface singularities: the rational quadruple points. We will determine the base space of the semi-universal deformation of such a singularity. The answer turns out to be unexpectedly simple: the isomorphism type of the base space of a rational quadruple point is completely determined by two numbers, s and n. The base space then is isomorphic to $S \times B(n)$, where S is a smooth germ of dimension s and where B(n) is the space defined by the set of equations (6.14). A rational quadruple point with the following star shaped resolution graph



has the factor B(n) in its base space. We call such a singularity an n-star. At the moment of writing, we have been unable to prove most of the following properties of the space B(n). It should not be too hard to settle the following

Conjecture (6.1) :

The space B(n) has the following properties:

1) $\mathrm{Embdim}(B(n)) = 5n - 1$.

2) B(n) has n+1 irreducible components Y_k , k=0, 1, . . . , n with $\dim(Y_k) = 2n-1+2k$.

3) $\mathrm{Mult}(Y_k) = \binom{n}{k}$, so only Y_0 and Y_n are smooth.

4) Y_k has a smooth normalization.

(Property 1) is trivial and only included for sake of completeness.) We know that (6.1) is true for n = 1 and n = 2 and we know that B(n) has, besides Y_0 and Y_n, at least n-1 other components.

In general there are several approaches to find the semi-universal deformation of a (normal surface) singularity $\tilde{X}$. In the first place there is the *direct method* : one starts with the set of equations defining $\tilde{X}$ as embedded in some high $\mathbb{C}^N$ and then one just computes. For this to work in practice the equations must have a sufficiently strong structure. For example rational triple points (see [Tj 2]) (Cohen-Macaulay codimension 2), the cone over the rational normal curve of degree n (see [Pi]), n lines in $\mathbb{C}^n$ etc, can be handled in this way. It seems however that the equations for the rational quadruple points are not known sufficiently well to compute the base spaces for them in this way. Secondly, there is the method of *(partial) resolutions.* Here one starts with a (partial) resolution Y of $\tilde{X}$ and then studies the deformation theory of Y (which is usually much simper) and finally one tries to blow down the deformed Y to get a deformation of $\tilde{X}$. This methods works quite well to get information on the components of the base space for rational singularities. For example, all deformations of a resolution of $\tilde{X}$ can be blown down and give rise to the so-called *Artin component* of (the base space of) $\tilde{X}$ (see [Wa1]). Recently, Kollar and Shepherd-Barron [K-S] developed a method by which one can, for instance, determine the number of components in the base space of a cyclic quotient singularity. (From their approach it is also clear that the n-star singularity has (at least) n+1 components in its base space.) However, the list of resolution graphs of rational quadruple points is quite long and contains many 'exceptional' graphs, so this method seems to be quite involved. Furthermore, it does not lead really to equations for the base spaces.

We propose to use a different method: the method of *projections.* Here one starts with $\tilde{X}$ embedded in some high $\mathbb{C}^N$ and then we project $\tilde{X}$ generically into $\mathbb{C}^3$. The image X then will have a curve Σ as double locus. By the theorems of § 4 the base space of admissible deformations of $\Sigma \hookrightarrow X$ is up to a smooth factor the same as the base space of $\tilde{X}$. Now essentially because Σ is Cohen-Macaulay of codimension 2 and

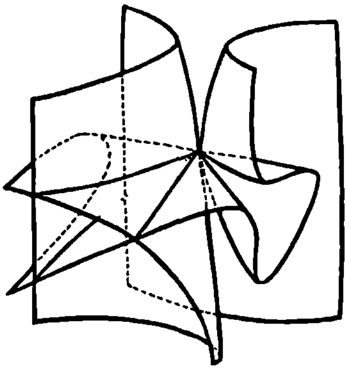
X is given by *one* equation, this is much more 'computable' then working directly with the equations for $\tilde{X}$. At first sight it seems that this method has two serious drawbacks. In the first place one has to choose a *generic* projection (to get an ordinary double curve) and naturally given projections usually are not generic. In the second place it is quite *hard* to find the explicit equation for X. For rational triple points it is already a lot of work to write down explicit equations for X corresponding to the different resolution graphs and for quadruple points it becomes quite hopeless. We only give one example of our (incomplete) list. (It appears that it is convenient to use the theory of *limits* (see [Str]) to obtain equations for singularities that come in series.)

Example (6.2) :

Equation :

$$f = (x-y).\left((x+y).(z^2+xy^2) + (x-y)^{k+1}.y^2\right) + z^l.(z^2+xy^2)^2$$

Qualitative picture of $X_{\mathbb{R}} := \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$:



Resolution graph of the normalization $\tilde{X}$:

So it seems that we are stuck again. But now it turns out that the first mentioned drawback can be turned into an advantage: every singularity has many generic projections with essentially distinct curves Σ. It turns out that *all* curves Σ that could *a priori* occur as double curves of projections of rational quadruple points actually do occur in projections of the special n-star singularities mentioned above. Because $\mathrm{Def}(\Sigma,X)$ mainly depends on the curve (in the sense of (3.27) and (3.28)) we get that the base space of any rational quadruple point is the same up to a smooth factor to the base space of a certain n-star singularity. So one has to compute only the base space for the n-star singularity and for this one can use a particular nice set of coordinates and in this way we find the equations for the spaces $B(n)$.

We start with some general numerical relations related to a generic projection.

Lemma (6.3) :

Let $\tilde{X} \subset \mathbb{C}^N$ be a ((multi-) germ of a) normal surface singularity, where $N = \mathrm{Embdim}(\tilde{X})$ is the embedding dimension of $\tilde{X}$. Let $L: \mathbb{C}^N \longrightarrow \mathbb{C}^3$ be a generic linear projection and let $X = L(\tilde{X}) \subset \mathbb{C}^3$ its image. Let Σ be the reduced singular locus of X and let $\tilde{H}$ and H be the generic hyperplane sections of $\tilde{X}$ and X respectively. Then one has:

i) $m := \mathrm{Mult}(\tilde{X}) = \mathrm{Mult}(X) = \mathrm{Mult}(\tilde{H}) = \mathrm{Mult}(H) \geq N-1$.

ii) $\mathrm{Mult}(\Sigma) = \delta(H) - \delta(\tilde{H})$.

iii) $\delta(H) \geq m.(m-1)/2$; $\delta(\tilde{H}) \geq m-1$.

iv) $\mathrm{type}(\Sigma) \geq N-3$.

proof : i) is obvious because we have a linear projection. The inequality expresses the minimality of the embedding of $\tilde{X}$ in $\mathbb{C}^N$. Statement ii) follows by moving the hyperplane H away from the special point. We then get as intersection with X a curve with $\mathrm{Mult}(\Sigma)$ ordinary double points. But the jump in δ in a family of curves is equal to the δ of the special fibre of the normalization of the family (see [L-L-T]), so in this case is equal to $\delta(\tilde{H})$. Statement iii) is a generality: given the embedding dimension and the multiplicity of the curve, one has a lower bound for its δ-invariant, which is in the stated cases as above. (exercise). Statement iv) follows the following: Σ is Cohen-Macaulay of codimension 2, so the equations for Σ are obtained as the maximal minors of an $t\times(t+1)$ matrix. Then type(Σ) = t. As in (5.1) this gives us an embedding of $\tilde{X}$ into a smooth space of dimension 3+t, hence $N \le t+3$. ⊠

Lemma (6.4) :

If $\tilde{X}$ is a germ of a rational surface singularity, then all the inequalities of (6.3) are in fact equalities.

proof : This lemma is a reflection of the strong minimality properties enjoyed by rational singularities. For the fact that $N = m + 1$ we refer to [Ar2]. For the statement that $\delta(\tilde{H})=m-1$ see([Str], 4.1.13). As now $\tilde{H}$ is a curve with minimal δ-invariant, the same is true for a generic projection H, hence $\delta(H) = m.(m-1)/2$ (we omit the proof). Then $\mathrm{Mult}(\Sigma) = (m-1).(m-2)/2$. This can also be seen directly: by a result of Karras (see [Ka]) every rational m-tuple point has a normally flat deformation to the cone over the rational normal curve of degree m. When we project this cone to $\mathbb{C}^3$ we get a cone over a rational curve in $\mathbb{P}^2$ of degree m. Such a curve has $(m-1).(m-2)/2$ double points. As clearly the multiplicity of Σ does not change under this deformation, Mult(Σ) has to have this value for all rational m-tuple points. The statement about the type can be seen as follows: because Σ is Cohen--Macaulay, the sub-scheme of $\mathbb{C}^2$ given by $\Sigma \cap H$ has length $(m-1).(m-2)/2$ and by (6.3) $\mathrm{type}(\Sigma\cap H) \ge m-2$. From these fact alone it already follows that the ideal of $\Sigma\cap H$ is the ideal $\mathfrak{m}^{m-1}$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{y,z\} = \mathcal{O}_{\mathbb{C}^2,0}$. Hence indeed $\mathrm{type}(\Sigma)=\mathrm{type}(\Sigma\cap H) = m-2$. ⊠

Corollary (6.5) :

$\tilde{X}$ rational triple point $\Rightarrow$ Σ is smooth, i.e. X is a line singularity.
$\tilde{X}$ rational quadruple point $\Rightarrow$ Σ has multiplicity 3 and type 2.

proof : Immediate from (6.4). ⊠

Lemma / Definition (6.6) :

Let Σ be a Cohen-Macaulay curve germ of multiplicity 3 and type 2. Then the equations for Σ can be obtained as the 2×2-minors of the following matrix:

$$M = \begin{pmatrix} y & z+a & b \\ c & y+d & z \end{pmatrix}$$

Here a, b, c and d are functions only depending on x. We define the λ-invariant of such a curve as:

$$\lambda(\Sigma) := \min(\mathrm{ord}(a), \mathrm{ord}(b), \mathrm{ord}(c), \mathrm{ord}(d))$$

Conversely, if $\lambda(\Sigma) \geq 1$, then the minors of the above matrix do define a Cohen-Macaulay curve germ of multiplicity 3 and type 2.

proof : Choose a generic projection of Σ on a line with coordinate x. Then Σ can be considered as the total space of a flat deformation of Σ intersected with x = 0. This sub-scheme of $\mathbb{C}^2$ is defined by m^2 = (y^2, yz, z^2). As these equations can be obtained from the matrix as above (with a=b=c=d=0), we find the indicated form for the equations of Σ. (A similar bigger matrix can be written down for the curves Σ appearing as double locus of a rational m-tuple point with m≥5.) ⊠

Remark (6.7) :

Curves of multiplicity 3 can be classified and J. Stevens has sent us the complete list. However, it turns out to possible to pursue our arguments without going into the fine structure of this classification.

Proposition (6.8) :

Let Σ and coordinates x, y, z as in (5.15). Let $I = (\Delta_1, \Delta_2, \Delta_3)$ the ideal of Σ defined by the minors of the matrix M. Consider the function

$$\Phi := \det(\tilde{M}) \in \mathbb{C}\{x\}[y,z] \, ; \qquad \tilde{M} := \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ y & z+a & b \\ c & y+d & z \end{pmatrix}$$

where $(\alpha_1, \alpha_2, \alpha_3) := x^{-\lambda}.(dy-cz,\ ad-bc,\ az-by)$, $\lambda = \lambda(\Sigma)$.

Then Φ has the following properties:

i) $\Phi \in \int I$.

ii) $\mathrm{Mult}(\Phi) = 3$; $\deg_{(y,z)}(\Phi) = 3$; $\Phi(0,y,z) \neq 0$.

iii) Consider a 3x3-matrix h with entries in $\mathbb{C}\{x,y,z\}$ with *generic constant part* h_0. Then the space X defined by $\Phi + h.\Delta.\Delta = 0$ has precisely Σ as singular locus, has a smooth normalization $\tilde{X}$ and the inverse image of $X \cap \{x=0\}$ on $\tilde{X}$ is a smooth curve.

proof : Let us first indicate the geometrical significance of a function Φ having properties i) and ii). The intersection of Σ with the plane $x=c$, $c \neq 0$ consists of three distinct points in the (y,z)-plane. Multiplying together the three linear factors describing the lines through the three pairs of points we get a polynomial Φ of degree 3 in y and z with coefficients depending on x. A direct computation then shows that Φ can be written as the above determinant. The Cramer matrix $\tilde{N}$ of 2x2-minors of $\tilde{M}$ is seen be to equal to

$$\tilde{N} = \begin{pmatrix} \Delta_1 & \alpha\Delta_2 + \beta\Delta_3 & \alpha\Delta_1 + \beta\Delta_2 \\ \Delta_2 & \gamma\Delta_1 + \delta\Delta_2 & \alpha\Delta_2 + \beta\Delta_3 \\ \Delta_3 & \gamma\Delta_2 + \delta\Delta_3 & \gamma\Delta_1 + \delta\Delta_2 \end{pmatrix}$$

(where $(\alpha,\beta,\gamma,\delta) = -x^{-\lambda}.(a,b,c,d)$), which shows that the matrix $\tilde{M}$ satisfies the rank condition, so indeed $\Phi \in \int I$. (see (3.31) and (4.12)). (We thank J. Stevens for pointing out to us that our Φ is equal to $\left(\Delta_1 . \Delta_3 - \Delta_2^2\right)/x^{\lambda}$.)

Now we turn to statement iii) of the proposition. The curve $X \cap \{x=0\}$ has an equation of the form $\Phi(0,y,z) + G(y,z)$, where G starts with a *generic* quartic in y and z, because of the genericity of h_0. From this it follows that $\delta(X \cap \{x=0\}) = 3$. For small values of c the curve $X \cap \{x = c\}$ has three ordinary double points, hence the $X \cap \{x = c\}$ is a δ-constant family of plane curves. Consequently, X has precisely Σ as singular locus, $\tilde{X}$ is smooth and the inverse image of $X \cap \{x=0\}$ on $\tilde{X}$ is smooth. ⊠

Corollary (6.9) :

In the above situation we have $\int I/I^2 \approx \mathbb{C}[x]/x^{\lambda}$, and a $\mathbb{C}$-basis for $\int I/I^2$ is given by $\Phi, x.\Phi, \ldots, x^{\lambda-1}.\Phi$.

proof : This can be checked by a direct calculation, but it is much nicer to apply here a beautiful theorem of D. Mond & R. Pellikaan (see [M-P], thm.4.4) which implies that for a weakly normal surface X in $\mathbb{C}^3$ with singular locus Σ and with a *Gorenstein* normalization $\tilde{X}$ the module $\int I/I^2$ is cyclic with generator the equation F of X and as annihilator the (t-1)x(t-1)-minors of the (t+1)×(t+1) matrix of the matrix factorization of F (as in (4.10)). In our case $\tilde{X}$ is smooth by (6.8)iii) and t=2, so the annihilator of $\int I/I^2$ is the ideal of of the entries of the matrix $\tilde{M}$ of (6.8), which is the ideal (y,z,x^{λ}). ⊠

Now we have sufficiently detailed information about the structure of the set of all weakly normal surfaces which have a curve Σ of multiplicity three (and type 2): they all have an equation of the form

$$F_{p,h}(\Sigma) := x^{p}.\Phi + (h.\Delta).\Delta$$

where Φ is as in (6.8) , h is a 3x3-matrix with entries in $\mathbb{C}\{x,y,z\}$. We let $X_{p,h}(\Sigma)$ be the surface germ defined by $F_{p,h}(\Sigma) = 0$. Note that by (6.9) $F_{p,h}(\Sigma) \in I^2$ if $p \geq \lambda(\Sigma)$.

Lemma (6.10) :

With the notation as above, let h be a matrix with generic constant part h_0. Then the tangent cone of the surface $X_{p,h}(\Sigma)$ is the cone over a curve $C \subset \mathbb{P}^2$, which has the following structure:

Case A: $\lambda(\Sigma) \geq 2$, $p \geq 2$; C consists of four distinct lines, all passing through a single common point.

Case B: $\lambda(\Sigma) \geq 2$, $p = 1$; C is an irreducible rational quartic curve with a unique singular point of type D_4, D_5 or E_6.

Case C: $\lambda(\Sigma) = 1$, $p = 1$; C is an irreducible rational quartic curve with one (A_5), two ($A_3 + A_1$) or three ($3.A_1$) singular points.

proof : If $p \geq 2$ then the tangent cone of $X_{p,h}(\Sigma)$ is determined by the term $(h.\Delta).\Delta$, because Φ has multiplicity 3. If $\lambda(\Sigma) \geq 2$, then the lowest order terms in the matrix M of (6.6) are the y and z, so for generic h we get as tangent cone a general quartic in y and z, which settles case A. If $p = 1$ and $\lambda(\Sigma) \geq 2$, then the lowest order term of $F_{p,h}$ contains also a term $x.\Phi$. Corresponding to the cases that $\Phi(0,y,z)$ is equivalent to $y^3 + z^3$, $y^2.z$, y^3 we then find a D_4, D_5 or an E_6 on C, which settles case B. The remaining case is $\lambda(\Sigma) = 1$, which is most involved. If we replace a, b, c, d in the matrix M of (6.6) by their linear parts, the minors of it define still a (possibly non-reduced) curve of multiplicity 3, which is the cone over a subscheme of $\mathbb{P}^2$ of length 3. One can check that an irreducible quartic C which contains such a subscheme in its singular locus has to have a total δ equal to 3, hence is rational. For generic h only the indicated cases do occur. ⊠

Theorem (6.11) :

For generic h and $1 \leq p \leq \lambda(\Sigma)$ the surface $X_{p,h}(\Sigma)$ has as normalization a p-star singularity.

proof : We blow up $\mathbb{C}^3$ at the origin. Let Σ' and X' be the strict transforms of Σ and $X_{p,h}(\Sigma)$. X' will have the tangent cone of $X_{p,h}(\Sigma)$ as exceptional divisor. If $p \geq 2$, then $\lambda(\Sigma) \geq 2$, so Σ' will still be a curve germ of multiplicity 3, and $\lambda(\Sigma')=\lambda(\Sigma)-1$, as one easily sees from blowing up the matrix M of (6.6). Also, by (6.10), the exceptional divisor of X' will consist of four lines through a point, which is also the singular point of Σ'. Around this point the surface X' will have a singularity of type $X_{p-1,h'}(\Sigma')$, as follows by looking at the equation $F_{p,h}$ in the x-chart. Because the tangent cone is reduced, X' will be smooth apart from this singularity. As only the constant part of h enters in the genericity assumption for (6.10) and the constant part of h' is the same as that of h, we can thus inductively go further with blowing up. After p-1 blow ups we have introduced four chains of rational curves of lenght p-1 and we are left with a singularity of type $X_{1,h''}(\Sigma'')$. Now there are two cases: $\lambda(\Sigma'') \geq 2$ and $\lambda(\Sigma'')=1$. These correspond to the cases B and C of (6.10). In each of these cases the tangent cone of $X_{1,h''}(\Sigma'')$ is an irreducible rational quartic curve. In the first case we find after one further blow up still a unique special point of type $X_{0,h'''}(\Sigma''')$, which has by (6.8) a *smooth* normalization (and the inverse inverse image of the quartic is also smooth). In the second case we get after blowing up $X_{1,h''}(\Sigma'')$ a surface X''' with singular locus Σ''' which can have one, two or three disjoint parts. We claim that X''' again has a smooth normalization and that the inverse image of the quartic is also smooth. This can be seen by applying the same idea as in the proof of (5.17)iii): around a part of Σ''' the germ of X''' can be considered as the total space of a family of curves with as special fibre the (germ of the) exceptional quartic. It is not hard to see that this is a family with constant δ (equal to 1, 2 or 3), which proves the claim. Our conclusion is that $X_{p,h}(\Sigma)$ for generic h has as normalization a singularity which has as resolution graph the graph of the p-star singularity. By keeping track of the order of vanishing of the function x along all exceptional curves, one can compute all the self- intersections and they are as for the p-star singularty. ⊠

Corollary (6.12) :

Let X be weakly normal surface in $\mathbb{C}^3$ with (reduced) singular locus Σ. Assume that Σ has multiplicity 3 and type 2. Let I be the ideal of Σ and f=0 the equation of X. Then the base space of the semi-universal deformation of the normalization $\tilde{X}$ of X is up to smooth factors the same as the base of the p-star singularity, where $p = \dim\left(\int I/(I^2 + (f))\right)$.

proof : As one has $\int I/I^2 \approx \mathbb{C}[x]/(x^\lambda)$ by (6.9), there are only a finite number of possibilities for $\int I/(I^2 + (f))$, which by (6.11) are all realized by projections of p-star singularities. Now apply (3.28), (4.3) and (4.4).

⊠

To complete the picture, we compute the hull of Def(Σ,X) for a particular nice projection of the n-star singularity.

Let the curve Σ_n be defined by the ideal

$$I = (\Delta_1, \Delta_2, \Delta_3) = (M_2.M_3,\ M_3.M_1,\ M_1.M_2)$$

where $M_i = L_i(y,z) + x^n$ and L_i are linear forms with $L_1+L_2+L_3 = 0$.

Lemma (6.13) :

A basis for T^1_Σ is given by the classes of the normal vectors $x^q.A_1$, $x^q.A_2$, $x^q.A_3$ (q=0,1, . . . , n-1) and $x^q.B$ (q=0,1, . . . , n-2) where the A_i and B are

$$\begin{array}{lll}
A_1 : (\Delta_1,\Delta_2,\Delta_3) & \longmapsto & (M_2 - M_3,\ 0,\ 0) \\
A_2 : \quad ,, & \longmapsto & (0,\ M_3 - M_1,\ 0) \\
A_3 : \quad ,, & \longmapsto & (0,\ 0,\ M_1 - M_2) \\
B : \quad ,, & \longmapsto & (M_2+M_3,\ M_3+M_1,\ M_1+M_2)
\end{array}$$

A basis of Tors(Ω^1_Σ) is given by the classes of the differentials

$$\left.\begin{array}{l}
3x^q\ M_1.d(M_2 - M_3), \\
3x^q\ M_2.d(M_3 - M_1), \\
3x^q\ M_3.d(M_1 - M_2)
\end{array}\right\} (q=0,\ldots,n-1).$$

proof: By a direct computation. ⊠

Consider the surface X_n defined by $F:= \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = 0$, which has as normalization $\tilde{X}_n$ an n-star singularity. We determine the semi-universal deformation of $\tilde{X}_n$ by computing essentially the semi-universal admissible deformation of $\Sigma_n \hookrightarrow X_n$. To formulate the result we need some more notation.

Definition (6.14) :

i) Let be given a natural number $n \geq 1$ and $P = \sum_{k\geq 0} p_k . x^k \in \mathbb{C}[x]$. We define the *bracket* $[P]$ as $[P] := \sum_{k\geq n} p_k . x^{k-n}$.

ii) Let $\mathbb{C}[a_1, a_2, a_3, e, b]$ the polynomial ring in the *coefficients* of polynomials $a_1, a_2, a_3, e \in \mathbb{C}[x]$ of degree n-1 and $b \in \mathbb{C}[x]$ of degree n-2. (So it is a polynomial ring in 5n-1 indeterminates and if n=1 there is no b.)

iii) Consider the following conditions on a_j, e and b :

$e.a_j - [e.a_j].b + [[e.a_j].b].b - [[[e.a_j].b].b].b + \ldots = 0 \bmod x^n$

(The dot is multiplication of polynomials and $j = 1, 2, 3$.)

Note that this a priori infinite series in fact breaks off, so these conditions indeed lead to polynomials in the coefficients of the a_j, e and b. Let J_n be the ideal in $\mathbb{C}[a_1, a_2, a_3, b, e]$ generated by these polynomials.

iv) Let $B(n) := \mathrm{Spec}(\mathbb{C}[a_1, a_2, a_3, b, e]/J_n)$.

Theorem (6.15) :

i) The base space C(n) of a semi-universal deformation of the n-star singularity $\tilde{X}_n$ is isomorphic to $B(n)\times \mathbb{C}$ $(n\geq 2)$ or $B(1)$ $(n=1)$. (The factor $\mathbb{C}$ corresponds to the cross-ratio of the four points on the central $\mathbb{P}^1$ in the resolution graph of the n-star.)

ii) A projection of a semi-universal family for $\tilde{X}_n$ over C(n) is given by the admissible family defined by the equation

$$\tilde{\Delta}.(\tilde{\Delta} + \alpha + \lambda.\beta) = 0$$

where (remember the vector notation and summation conventions.):

* $\tilde{\Delta} = \Delta + \sum_{i=1}^{3} a_i.A_i + b.B + \rho$.

  a_i and b as in (6.14)ii).

  A_i and B as in (6.13)i).

  $\rho = (a_4^2 + a_1.a_2 + a_2.a_3 + a_3.a_1).(1,1,1)$

  ($\tilde{\Delta}$ describes the semi-universal deformation of Σ_n.)

* $\alpha = \sum_{i=1}^{\infty} \alpha_i$.

  $\alpha_1 = e.(M_1, M_2, M_3)$.

  $\alpha_2 = e.b.(1,1,1) + \sum_{i=1}^{3} 2.e.a_i.V_i - \sum_{i=1}^{3} [e.a_i].W_i$.

  $\alpha_3 = [e.(a_1.a_2 + a_2.a_3 + a_3.a_1)](1,1,1) - \sum_{i=1}^{3} 2.[e.a_i].b.V_i +$
  $+ \sum_{i=1}^{3} [[e.a_i].b].W_i$.

  $\alpha_{i+1} = -[\alpha_i.b]$ for $i \geq 3$.

  e and b as in (6.14)ii).

  $V_1 = (0,-1,1)$, $V_2 = (1,0,-1)$, $V_3 = (-1,1,0)$.

  $W_1 = (0,-(M_1+M_2), M_1+M_3)$, $W_2 = (M_1+M_2, 0, -(M_2+M_3))$,

  $W_3 = (-(M_1+M_3), M_2+M_3, 0)$.

* The a_i, e and b satisfy the equations for B(n) (6.14)iv).

* $\lambda \in \mathbb{C}$ is the equisingular parameter (absent if n=1).

  $\beta = (M_1 M_2, 0, 0)$.

We omit the proof, which consists of many pages of straightforward but very tedious computations. Especially we would like to thank J. de Jong for his enthousiastic persistence to complete the computation. A first step in the computation consists of writing down an admissible first order deformation which gives precisely T^1_X . Then one proceeds along the lines of (3.23). (Note that example (3.24) is the same as the above one with n=1.) It is our experience that the bases chosen in (5.22) are quite convenient, but there might be better ones. In the deformation of Σ_n there appear at most quadratic terms. This one of the reason that after three steps an inductive pattern for the α_i emerges. ⊠

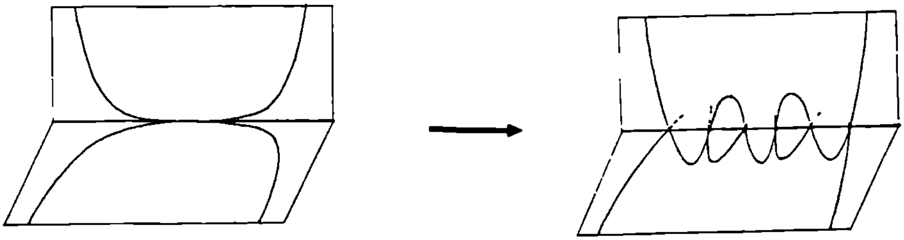
Remark (6.16) :

It is easy to see two of the components of the space $B(n)$ The first one is obtained by putting $e=0$. Then the a_i and b can be arbitrary, so it has dimension $4n-1$. This corresponds to deformations 'which stay in I^2' and gives rise to the Artin component. A second component is obtained by putting the a_i equal to zero and thus has dimension $2n-1$. These deformations include the ones for which Σ_n is not deformed at all. Note further that the equations for $B(n)$ are *linear* in the a_i. We expect that the space $B(n)$ is flat over the b-parameter. For $b=0$ the equations for $B(n)$ define linear spaces with certain multiplicities. These facts suggest the truth of conjecture (6.1).

We conclude this paragraph by giving some adjacencies for rational quadruple points which are quite easy to see when projected into $\mathbb{C}^3$, but which are, so far as we know, not so easy to see in an other way.

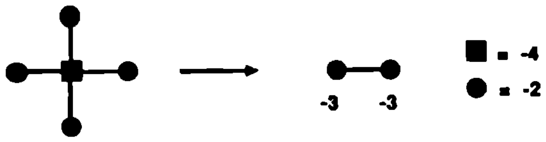
Examples (6.17) :

i) The admissible deformation of $\Sigma_n \hookrightarrow X_n$ described by the equation $(\Delta + t.B + t^2(1,1,1))^2 = 0$, gives for generic t values a surface which has as normalization a surface with n singular points isomorphic the cone over the rational normal curve of degree 4. Indeed, the deformation of the singular locus looks like:



We can perform an additional deformation of this surface in such a way that at p of those n special points we get a triple point whereas at the other n-p points we smooth out the curve. This leads to deformations of the n-star singularity having precisely p triple points $(0 \le p \le n)$. Hence $B(n)$ has at least n+1 components.

ii) J. Wahl describes the following deformation [Wa1]:



When projected to $\mathbb{C}^3$, it can be realized by the admissible deformation given by the equation

$$f_t = (yz)^2 + (y(y-x^2+tx))^2 + (z(z-x^2+tx))^2 + 2.txyz(y+z-x^2+tx) = 0.$$

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NIET GEISOLEERDE HYPEROPPERVLAK SINGULARITEITEN

SAMENVATTING

Dit proefschrift is een voortzetting van de studie van niet geïsoleerde singulariteiten. In hoofdstuk één bestuderen we hyperoppervlak singulariteiten met een gladde ééndimensionale singuliere locus. Het doel is het homotopy type van de Milnorvezel te bepalen. Dit wordt gedaan door deformaties te bekijken die de singuliere locus vasthouden, èn zodat op een algemeen punt van de singuliere locus de geïnduceerde deformatie triviaal. Door deformaties op te schrijven zodat in een algemene vezel alleen maar 'eenvoudige' singulariteiten optreden, kan men de Milnorvezel bepalen door de Milnorvezels van de 'eenvoudige' singulariteiten aan elkaar te plakken. Door technische problemen moeten we ons beperken tot bepaalde klasses van singulariteiten met een gladde ééndimensionale singuliere locus.

In hoofdstuk twee wordt het virtuele aantal D_∞ punten ingevoerd voor hyperoppervlakken met een ééndimensionale singuliere locus en transversaal A_1 singulariteiten. Dit is een generalisatie van het aantal D_∞ punten dat voorkomt als men de functie algemeen genoeg verstoort (waarbij de singuliere locus plat deformeert) indien de singuliere locus een volledige doorsnijding is. De 'continuiteit' van het virtuele aantal D_∞ punten wordt bewezen, alsmede een formule voor de Euler karakteristiek van de Milnorvezel. Verder wordt aangetoond dat het totale virtuele aantal D_∞ punten van een compacte divisor met ééndimensionale singuliere locus en transversaal A_1 singulariteiten in een gladde variëteit een berekenbare globale invariant is.

In hoofdstuk drie wordt de formele 'toegestane' deformatie theorie van niet geïsoleerde singulariteiten ontwikkeld. Verder wordt er een verband gelegd met de deformatietheorie van normale oppervlak singulariteiten. Toepassingen bestaan uit een formule voor de dimensie van vergladdingscomponenten van normale oppervlak singulariteiten, alsmede bepaling van basisruimtes van semi-universele deformaties van rationale quadrupel punten. Ook worden enkele generalisaties van formules van R. Pellikaan bewezen.

CURRICULUM VITAE

Ik ben geboren op 16 februari 1962 te Reeuwijk. Van 1968 tot 1974 bezocht ik de St. Jozefschool te Reeuwijk-dorp, en vervolgens was ik leerling aan het St. Antoniuscollege te Gouda, waar ik in 1980 het V.W.O. diploma behaalde. In 1980 begon ik natuurkunde te studeren in Leiden, maar na een half jaar besloot ik om te schakelen naar de wiskundestudie. Het kandidaatsexamen werd in december 1982 afgelegd, en het doctoraalexamen in augustus 1984 (cum laude). In september 1984 trad ik in dienst bij Z.W.O., om bij Prof. dr. J.H.M. Steenbrink promotieonderzoek te doen in Leiden in het kader van het interuniversitaire onderzoeksproject 'singulariteitentheorie'. Van juli 1986 tot april 1987 heb ik buitengewoon verlof gekregen om aan de universiteit van Kaiserslautern (B.R.D.) onder leiding van Prof. dr. G.-M. Greuel te werken. Sinds 1 januari 1988 is mijn standplaats Nijmegen geweest.

Stellingen behorende bij het proefschrift
non-isolated hypersurface singularities
van Theo de Jong.

1. Varchenko's afschatting van het aantal gewone dubbelpunten op projectieve hyperoppervlakken kan bewezen worden zonder gebruik te maken van de semicontinuïteit van het spectrum.

A.N. Varchenko: On semicontinuity of the spectrum and an upper bound for the number of singular points of projective hypersurfaces. Soviet Math Dokl. 27 (1983) 735-739.

2. Laat $f \in \mathbb{C}\{x,y,z\}$ een ééndimensionale singuliere locus hebben en transversaal type A_1. Laat I het ideaal zijn dat de gereduceerde singuliere locus definieert. Neem aan dat de singuliere locus geen volledige doorsnijding is en dat $f \in I^2$. Dan is f irreducibel.

3. Het rationale tripelpunt $A_{k,k,k}$ heeft een éénparameter deformatie zodat in een generieke vezel k+1 $A_{0,0,0}$ singulariteiten voorkomen. Bovendien kunnen in een generieke vezel van zo'n deformatie geen andere singulariteiten optreden.

G. Tjurina: Absolute isolatedness of rational singularities and triple rational points. Funct. Anal. Appl. 2 (1968) 324-333.

4. Laat $\varphi\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ een kiem van een eindig bepaalde quasihomogene holomorfe afbeelding zijn en φ_t een generieke verstoring van φ. Zij X_t een goede representant van $\varphi_t(\mathbb{C}^2)$. Dan geldt: $\mathrm{cod}(\mathcal{A}_e,\varphi) = \chi(X_t) - 1$, waarbij $\mathrm{cod}(\mathcal{A}_e,\varphi)$ de codimensie van φ met betrekking tot links-rechts equivalentie is en χ de topologische Euler karakteristiek.

T. de Jong en G.R. Pellikaan: werk in voorbereiding.

5. De landbouwsanering samen met de nieuwe maatregelen met betrekking tot de studiefinanciering maken het voor boerenzonen c.q. boerendochters wel erg moeilijk om te gaan studeren.

6. Het moet mogelijk zijn aspecten van de economie uit te leggen zonder daarbij beledigend op te treden. De paragraaf " De varkenscyclus" in het boek "De kern van de economie, deel II", waarin Prof. A. Heertje doet voorkomen dat alle varkenshouders dom zijn, dient dan ook herschreven te worden.

A. Heertje: De kern van de economie dl II. Vierde druk. Stenfert Kroese, Leiden 1977.

7. Het is verwonderlijk dat er mensen zijn die trots zijn op het feit dat ze niet weten wat een priemgetal is.

8. Het zal Jos Brink niet lukken veertien moeders van veertien kinderen in zijn programma "Wedden dat" te krijgen.